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ON FREDHOLM PROPERTIES OF TOEPLITZ OPERATORS IN BERGMAN SPACES

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ABSTRACT. We consider Toepliz operators with integrable symbols acting on Bergman spaces A^p, $1 < p < \infty$, of the open unit disc of the complex plane. We combine some of the best known results on compactness of Toeplitz and Hankel operators in order to generalize the results on Fredholm properties of Toeplitz operators. We pay special attention to some concrete examples.

1. INTRODUCTION AND PRELIMINARIES.

The theory of Toeplitz operators in Bergman spaces have found applications in areas like deformation quantization of theoretical physics, which relates to geometric structures of classical mechanics their algebraic quantum counterparts, see [1], [2], and to soliton theory, [3], [4], where these operator theoretic methods can be applied to relax the standard assumptions on initial conditions of the KdV-equation. A Toeplitz operator T_a is determined by its symbol a, which is usually an integrable function on the underlying analytic manifold Ω. However, even in the simplest case when Ω is the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, not all operator theoretic properties of T_a following from the properties of the symbol are adequately understood: apparently, unnecessarily strong a priori assumptions on the regularity of a have been needed for many existing resuls.

In this paper we concentrate on relaxing the conditions for the Fredholm property of the operator T_a. Recall that a bounded operator T in a Banach space X is Fredholm, if it has an inverse S modulo compact operators, i.e. the $TS-I$ and $ST-I$ are compact in X. (We recall the details of the definitions and basic statements at the end of this section.) As well known, the Fredholm property is crucial in the theory of solutions of integral equations, which is in fact historically the starting point to the study of this operator class.

Consider the Banach space $L^p := (L^p(\mathbb{D}, dA), \| \cdot \|_p)$, where $1 < p < \infty$ and $dA = \pi^{-1} r dr d\theta$ is the normalized area measure on $\mathbb{D}$, and denote the Bergman space by A^p, which is the closed subspace of L^p consisting of analytic functions. The Bergman projection P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pf(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta)$$

for $z \in \mathbb{D}$. It is also known to be a bounded projection of L^p onto A^p for every $1 < p < \infty$. We denote the complementary projection of P by $Q := I - P$, where I is the identity operator.

For an integrable function $a : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions f, the Toeplitz operator T_a with symbol a is defined by

$$(1.1) \qquad T_a f(z) = P(af)(z) = \int_{\mathbb{D}} \frac{a(\zeta)f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta)$$

for $z \in \mathbb{D}$. By posing suitable sufficient conditions on the symbol a, the operator T_a extends as a bounded operator $A^p \to A^p$, see e.g. [5], [6]. This is trivially true for bounded a. The corresponding Hankel operator $H_a : A^p \to L^p$ is defined by $H_a = M_a - T_a$, where $M_a : A^p \to L^p$ is the pointwise multiplier $M_a : f \mapsto af$. Hence, we have

$$(1.2) \qquad H_b f(z) = \int_{\mathbb{D}} \frac{(b(z) - b(\zeta))f(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta)$$

for $z \in \mathbb{D}$.

In this paper we deal with the Fredholm properties of Toeplitz operators $T_a : A^p \to A^p$ both theoretically and by means of concrete examples. In Theorem 2.2, which is our main result, we give a sufficient condition for T_a to be Fredholm for a wide class of symbols a in L^1. While the proof is in a sense rather straightforward, the novelty is to combine the most general known results on the compactness of Toeplitz and Hankel operators with the standard Fredholm theory to obtain sufficient conditions for T_a to be Fredholm.

For a given symbol a, the study of the compactness of the Hankel operator $T_{1/a}$ is closely related with this analysis. We recall the following result of Luecking [7] (see also Theorem 8.35 of [6]; see below for the unexplained notation).

Theorem 1.1. *The following conditions are equivalent for all $b \in L^2$.*
(i) $H_b : A^2 \to L^2$ is compact.
(ii) We have

$$\|f \circ \varphi_a - P(f \circ \varphi_a)\|_2 \to 0 \ as \ |a| \to 1.$$

(iii) The function F_b defined by

$$F_b(z) = \inf_{h \in A^2} J_{b-h}(z) = \inf_{h \in A^2} \frac{1}{|D(z,1)|} \int_{D(z,1)} |b - h|^2 dA$$

belongs to $C_0(\mathbb{D})$, that is, $F_b(z) \to 0$ as $|z| \to 1$.
(iv) The function b can be decomposed as $b = b_1 + b_2$, where $b_1 \in C^1(\mathbb{D})$ with

$$(1 - |z|^2)\overline{\partial}b_1(z) \in C_0(\mathbb{D}),$$

and the function J_{b_2} belongs to $C_0(\mathbb{D})$.

For bounded symbols, a similar characterization of compact Hankel operators on A^2 was first given in [8, 9], which answered the question raised by Axler, [10]. This was recently extended to all Bergman spaces A^p with $1 < p < \infty$ in [11], where it was also shown that compactness of Hankel operators with bounded symbols is in fact independent of p. See [11] for further details about compact Hankel operators

with bounded symbols. For other recent works on Hankel operators in Bergman spaces we refer to [12] and [13].

In the next section, we will present a modification of the implication $(iv) \Rightarrow (i)$ of Theorem 1.1, which works for all Bergman spaces A^p and all symbols in L^1. It is worth noting that compactness criteria for $H_{1/a}$ often involve VMO_δ-type conditions (see Definition 1.4 below) and these are usually used to prove Fredholmness of T_a; see, e.g. [14, 11, 15], which show that if $a \in L^\infty \cap VMO_\delta^1$, then T_a is Fredholm on A^p if and only if $\tilde{a}$ is bounded away from zero in some annulus $\{z : r < |z| < 1\}$, where $\tilde{a}$ is the Berezin transform of a (see [6] for the definition of $\tilde{a}$). As remarked very recently in [16], Fredholmness of T_a can be characterized for $a \in BMO_\delta^1$ provided that $\tilde{a} \in L^\infty \cap VO$ by writing $a = \tilde{a} + (a - \tilde{a})$ and observing that $T_{a-\tilde{a}}$ is compact, so that T_a is Fredholm if and only if $T_{\tilde{a}}$ is Fredholm; then, one uses the results of [17] on the Fredholm properties of the operator $T_{\tilde{a}}$.

This naturally leads to the question of describing the Fredholm properties of Toeplitz operators with bounded symbols that are not in VMO_δ^1. In view of the weak condition for the compactness of T_a in [5], the requirement of a or $1/a$ to be in VMO_δ is certainly not necessary for T_a to be Fredholm. Finding a complete description remains as a considerable challenge, but we think that Theorem 2.2 and the examples we provide in Section 3 may well be useful for further studies.

The following notation will be used throughout the paper. By C, C', c etc. we denote positive constants, the exact value of which may vary from place to place but not in the same chain of inequalities. If the constant depends on some parameter or function, say, n or g, this is denoted by C_n or C_g etc. All function spaces consist of functions on the disc $\mathbb{D}$, unless otherwise stated. In particular, the space of bounded analytic functions on the disc is denoted by H^∞. For clarity, we write $C^k(\mathbb{D})$ for the space of k times continuously differentiable functions in $\mathbb{D}$, where $k \in \mathbb{N} \cup \{\infty\}$. The symbol $C_0^k(\mathbb{D})$ denotes the subspace consisting of functions with compact support in $\mathbb{D}$; k is omitted from the notation, if it equals 0. Given $f \in C^1(\mathbb{D})$, we denote the Wirtinger symbols by $\partial = \partial_x + i\partial_y$, $\overline{\partial} = \partial_x - i\partial_y$, where $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ for $z = x + iy$. Also, we use $\partial_r = \partial/\partial r$ and $\partial_\theta = \partial/\partial \theta$ for the polar coordinates. We denote the Möbius transform by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z} \quad \text{for } a, z \in \mathbb{D}.$$

For $0 < \rho < 1$ we set $\mathbb{D}_\rho := \{z \in \mathbb{D} : |z| \le \rho\}$. Given $a \in L^1$, we denote by a_ρ the function, which coincides with a on $\mathbb{D}_\rho$ and equals 0 elsewhere.

We let $\mathcal{D}$ be a family of the sets $D := D(r, \theta) \subset \mathbb{D}$, where

$$(1.3) \qquad D = \{\rho e^{i\phi} \mid r \le \rho \le 1 - \frac{1}{2}(1 - r) , \ \theta \le \phi \le \theta + \pi(1 - r)\}$$

for all $0 < r < 1$, $\theta \in [0, 2\pi]$. We denote $|D| := \int_D dA$ and, for $\zeta = \rho e^{i\phi} \in D(r, \theta)$,

$$\hat{a}_D(\zeta) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho \, d\varphi \, d\varrho.$$

In the sequel we will consider functions $a \in L^1$ such that there exists a constant $C > 0$ such that

$$(1.4) \qquad |\hat{a}_D(\zeta)| \le C$$

for all $D \in \mathcal{D}$ and all $\zeta \in D$. The following result is contained in Theorem 2.1 of [18], and it contains the generalized definition of the Toeplitz operator, which we use for symbols satisfying (1.4) and which coincides with the conventional definition whenever the integral (1.1) converges. This generalization of the definition of Toeplitz operators is thus a most natural one, it coincides with the conventional definition whenever the integral formula (1.1) makes sense, and it is considered from many points of view in [18], [19].

Theorem 1.2. *Let* $1 < p < \infty$. *If a symbol* $a \in L^1$ *satisfies the condition* (1.4), *then the limit*

$$(1.5) \qquad T_a f = \lim_{\rho \to 1} T_{a_\rho} f \ , \quad \text{where} \ \ T_{a_\rho} f(z) = \int\limits_{\mathbb{D}_\rho} \frac{a(\zeta) f(\zeta)}{(1 - z\bar\zeta)^2} dA(\zeta) \ , \quad f \in A^p,$$

converges in the strong operator topology and defines a bounded operator $T_a : A^p \to A^p$ *for all* $1 < p < \infty$.

Moreover, the transposed operator $T_a^* : A^q \to A^q$ *can be written as*

$$(1.6) \qquad\qquad T_a^* f(z) = \lim_{\rho \to 1} \int\limits_{\mathbb{D}_\rho} \frac{\overline{a}(\zeta) f(\zeta)}{(1 - z\bar\zeta)^2} dA(\zeta)$$

for $f \in A^q$ *and this limit also converges in the strong operator topology.*

Notice that for a fixed $0 < \rho < 1$ the restriction of any $f \in A^p$ to $\mathbb{D}_\rho$ is a bounded function and the operator T_{a_ρ} is bounded in A^p. Formula (1.5) allows us to define the Toeplitz operator even in many cases, where the defining integral of the conventional formula (1.1) does not converge. In addition, condition (1.4) is also necessary for the boundedness when a is positive. Whether our result gives a complete description of bounded Toeplitz operators with general symbols remains an open problem.

As for the compactness of Toeplitz operators, we have the following result, which is Theorem 2.6 of [5].

Theorem 1.3. *Let* p *and* a *be as in Theorem 1.2, in particular,* a *satisfies* (1.4). *If* a *has the property*

$$(1.7) \qquad\qquad \lim_{d(D) \to 0} \sup_{\zeta \in D} |\hat{a}_D(\zeta)| = 0,$$

where

$$d(D) = \text{dist}(D, \partial\mathbb{D}) = \inf\{|z - w| \ : \ z \in D, |w| = 1\},$$

then $T_a : A^p \to A^p$ *is compact.*

We next recall the definition of the space VMO_δ^p. Let us fix r for a moment. The notion of the mean oscillation $MO_r^p(f)$ of a function f in L^p is defined by

$$(1.8) \qquad MO_r^p(f)(z) = \left(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} |f(\zeta) - \hat{f}_r(z)|^p dA(\zeta) \right)^{1/p},$$

where $B(z, r)$ is the disc with center z and radius $r > 0$ in the Bergman metric (see [6], Section 8.1) and

$$(1.9) \qquad\qquad \hat{f}_r(z) = \frac{1}{|B(z,r)|} \int\limits_{B(z,r)} f \, dA.$$

We write $B(z) = B(z, 1)$ and $\hat{f} = \hat{f}_1$ in the sequel.

Definition 1.4. *The space of bounded mean oscillation BMO_r^p consists of all locally L^p-integrable functions for which $\sup_{z \in \mathbb{D}} MO_r^p(f)(z) < \infty$. If, in addition, $MO_r^p(f)(z) \to 0$ as $|z| \to 1$, we say that f is in VMO_r^p.*

The definition of the spaces is independent of r so that we write BMO_∂^p for BMO_r^p and VMO_∂^p for VMO_r^p. Note that $BMO_r^\partial \subset BMO_\partial^q$ properly for $q < p$, and $L^\infty \cap VMO_\delta^p = L^\infty \cap VMO_\delta^1$ for all $p > 1$.

We finish this section with a short account of Fredholm theory; for a full treatment, see, e.g., Section 3.16 of [20]. A bounded linear operator T on a Banach space X is said to be Fredholm if both its kernel and cokernel are finite dimensional. We note that Fredholm operators have closed range. The index $\operatorname{ind} T$ of T is defined to be the difference of the two dimensions.

We need the following characterizations. A bounded linear operator S is called a left parametrix (or a regularizer) of an operator T if $ST = I + K_1$ and a right parametrix if $TS = I + K_2$ for some compact operators K_1 and K_2. It is well known that a bounded linear operator T is Fredholm on X if and only if T has a left and right parametrix. Another useful characterization is that T is Fredholm if and only if the equivalence class $T + K(X)$ is invertible in the Calkin algebra $B(X)/K(X)$, the quotient of the spaces of bounded and compact operators on X. Also, we recall that the Fredholm index is stable under small perturbations; that is, for each operator $T \in B(X)$, there is an $\epsilon > 0$ such that $T + S$ is Fredholm and $\operatorname{ind}(T + S) = \operatorname{ind} T$, if $S \in B(X)$ and the operator norm satisfies $\|S\| < \epsilon$. Regarding compact perturbations, if T is Fredholm, then $T + K$ is Fredholm and $\operatorname{ind}(T + K) = \operatorname{ind} T$ whenever K is compact. In particular, if T is invertible and K is compact, then $T + K$ is Fredholm of index zero.

2. Sufficient condition for Fredholmness.

We start by the following generalization of one of the statements of Theorem 1.1. The proof follows the original one, [7], [6], but we have to pay attention to a number of details, because of the more general setting. We expose the details for the convenience of the reader.

Lemma 2.1. *Assume that $b \in L^1 \cap C^1(\mathbb{D})$ and that b satisfies the condition (1.4) and*

$$(2.1) \qquad\qquad (1 - |z|^2)\overline{\partial}b(z) \in C_0(\mathbb{D}).$$

Then, the Hankel operator $H_b : A^p \to L^p$ is compact for any $1 < p < \infty$.

The definition of the Hankel operator needs to be clarified, since the assumptions do not imply that M_b maps A^p into L^p; notice that by assumption the Toeplitz operator T_b is bounded in A^p at least in the sense of the generalized definition of Theorem 1.2. However, the multiplier maps for example H^∞ into L^1. Thus, arguments simpler than those in (2.3)–(2.5) and (2.11)–(2.16), below, can be used to obtain the norm bound

$$(2.2) \qquad\qquad \|H_b g\|_p \le C \|g\|_p$$

for all $g \in H^\infty$ so that the Hankel operator can be defined in the standard way by a bounded extension from the dense subspace H^∞ of A^p.

Proof. We start by the remark that

(2.3) $\langle T_b g, \partial h \rangle = 0, \quad \text{hence}, \quad \langle H_b g, \partial h \rangle = \langle bg - T_b g, \partial h \rangle = \langle bg, \partial h \rangle$

for all $g \in H^\infty$, b as in the assumption of the lemma, and $h \in C_0^\infty(\mathbb{D})$; note that the duality bracket is certainly well defined since $T_b g$ and bg belong to L^1. Indeed, following the proof of Havin's lemma (Lemma 8.30 in [6]; see in particular the formula above (8.8) there) we find that the Green theorem and the product rule imply for all $\varphi \in C^1(\mathbb{D})$

(2.4) $\displaystyle\int_{\mathbb{D}} \overline{\partial}(\varphi \overline{h}) \, dA = 0, \quad \text{hence}, \quad \int_{\mathbb{D}} \varphi \overline{\partial h} \, dA = - \int_{\mathbb{D}} (\overline{\partial}\varphi)\overline{h} \, dA.$

In particular,

(2.5) $\displaystyle\int_{\mathbb{D}} T_b g \, \overline{\partial h} \, dA = - \int_{\mathbb{D}} (\overline{\partial} T_b g)\overline{h} \, dA$

This vanishes, since $T_b g$ is analytic, and (2.3) thus holds true.

Second, we note that Havin's lemma holds also in the non-Hilbert setting as follows: we denote by X the closure of the subspace $\{\partial\psi : \psi \in C_0^\infty(\mathbb{D})\}$ in L^q, where $1/q + 1/p = 1$, and claim that

(2.6) $X = Q(L^q) = (I - P)(L^q).$

Indeed, we have for all $f \in L^p$, $\psi \in C_0^\infty(\mathbb{D})$, by (2.4)

(2.7) $\langle f, P\partial\psi \rangle = \langle Pf, \partial\psi \rangle = \langle \overline{\partial} Pf, \psi \rangle = 0 \ \Rightarrow \ P\partial\psi = 0 \ \Rightarrow \ X \subset Q(L^q).$

On the other hand, if $Q(L^q) \setminus X \neq \{0\}$, we fix a function $f \in Q(L^q) \setminus X$, $f \neq 0$ and apply the Hahn-Banach theorem in the dual pair of L^p and L^q to find a function $g \in L^p$, $g \neq 0$, such that

(2.8) $\langle g, f \rangle \neq 0 \ \text{ and } \ 0 = \langle g, \partial\psi \rangle = \langle \overline{\partial} g, \psi \rangle \ \forall \ \psi \in C_0^\infty.$

But these imply

$$\langle Qg, f \rangle = \langle g, Qf \rangle = \langle g, f \rangle \neq 0$$

and, since $X \subset Q(L^q)$,

$$\langle \overline{\partial}(Qg), \psi \rangle = \langle Qg, \partial\psi \rangle = \langle g, Q(\partial\psi) \rangle = \langle g, \partial\psi \rangle = 0 \ \forall \ \psi \in C_0^\infty.$$

so that $Qg \neq 0$ and $\overline{\partial}(Qg) = 0$, hence, Qg is a non-zero analytic function, which is a contradiction. Thus (2.6) holds.

We will need the following consequence of (2.6): for all $g \in L^p$ we obtain

$$\|Qg\|_p = \sup_{\substack{\tilde{h}\in L^q, \\ \|\tilde{h}\|_q \leq 1}} \langle Qg, \tilde{h} \rangle = \sup_{\substack{\tilde{h}\in L^q, \\ \|\tilde{h}\|_q \leq 1}} \langle Q^2 g, \tilde{h} \rangle = \sup_{\substack{\tilde{h}\in L^q, \\ \|\tilde{h}\|_q \leq 1}} \langle Qg, Q\tilde{h} \rangle$$

(2.9) $\displaystyle \leq \sup_{\substack{f\in X, \\ \|f\|_q \leq 1}} \langle Qg, f \rangle = \sup_{\substack{\tilde{f}\in L^q, \\ \|\partial\tilde{f}\|_q \leq 1}} \langle Qg, \partial\tilde{f} \rangle,$

where we used the fact that $\|Q\varphi\|_q \leq C\|\varphi\|_q$ for all $\varphi \in L^q$, since Q is a bounded projection.

We consider a sequence $(g_n)_{n=1}^\infty \subset A^p$, which is bounded in the norm of L^p and converges to 0 uniformly on compact subsets of $\mathbb{D}$. It suffices to show that H_b maps

this sequence into one converging to 0 in the norm of L^p. First, we choose for all n a polynomial $\tilde{g}_n$ such that

$$\|g_n - \tilde{g}_n\|_p \leq \frac{1}{n}. \tag{2.10}$$

Since T_b is a bounded operator by assumption, $T_b g_n \to 0$ as $n \to \infty$ in A^p if and only if $T_b \tilde{g}_n \to 0$ in A^p. Also, the polynomials $\tilde{g}_n$ converge to 0 uniformly on bounded sets and certainly also form a bounded set, as a consequence of (2.10). Thus, we may assume from the very beginning that the original sequence consists of polynomials, which in particular are members of $H^\infty \subset A^2$.

Given an arbitrary $\varepsilon > 0$ we fix a compact set K such that

$$(1 - |z|^2)|\overline{\partial}b(z)| < \varepsilon \quad \text{for all } z \in \mathbb{D} \setminus K \tag{2.11}$$

and then $N \in \mathbb{N}$ such that

$$|g_n(z)| \leq \frac{\varepsilon}{1 + \sup_{z \in K} |\overline{\partial}b(z)|} \quad \text{for all } z \in K, \ n \geq N. \tag{2.12}$$

We write for all $h \in \mathbb{C}_0^\infty$, using (2.3) and (2.4),

$$\langle H_b g_n, \partial h \rangle = \langle b g_n, \partial h \rangle = \int_{\mathbb{D}} b g_n \overline{\partial h} \, dA$$

$$= -\int_{\mathbb{D}} (\overline{\partial}(b g_n)) \overline{h} \, dA = -\int_{\mathbb{D}} (\overline{\partial}b) g_n \overline{h} \, dA. \tag{2.13}$$

Hence, by the Hölder inequality with $1/p + 1/q = 1$

$$|\langle H_b g_n, \partial h \rangle|$$
$$\leq \left(\int_{\mathbb{D}} (1 - |z|^2)^p |\overline{\partial}b(z)|^p |g_n(z)|^p dA(z) \right)^{1/p} \left\| \frac{h(z)}{1 - |z|^2} \right\|_q \tag{2.14}$$

Here we recall that $|\nabla h(z)|^2 = 2(|\partial h(z)|^2 + |\overline{\partial}h(z)|^2$ for all $z \in \mathbb{D}$ and thus obtain by Lemmas 8.31 (or its straightforward generalization) and 8.32 of [6]

$$\left\| \frac{h(z)}{1 - |z|^2} \right\|_q \leq C \|\partial h\|_q. \tag{2.15}$$

Moreover, we divide the L^p-integration in (2.14) into integrals over K and its complement, and obtain by (2.11), (2.12) for $n \geq N$ the bound

$$\int_{\mathbb{D}} (1 - |z|^2)^p |\overline{\partial}b(z)|^p |g_n(z)|^p dA(z)$$

$$\leq \sup_{z \in K} |\overline{\partial}b(z)||g_n(z)| + \int_{\mathbb{D} \setminus K} \left(\sup_{w \in \mathbb{D} \setminus K} (1 - |w|^2)^p |\overline{\partial}b(w)|^p \right) |g_n(z)|^p dA(z)$$

$$\leq \varepsilon + \varepsilon^p \|g_n\|_p^p \tag{2.16}$$

Combining this with (2.14), (2.15) and taking into account (2.9) yield $\|H_b g_n\|_p \leq C\varepsilon$, which was to be proven. $\square$

The following is our main result on the Fredholm properties of Toeplitz operators. As for the calculation of the Fredholm index, we comment it only in the examples in Section 3.

Theorem 2.2. *Assume that the symbol $a \in L^1$ has property* (1.4) *and that there is a function $b \in L^1$ satisfying* (1.4) *such that*
1°. *b and $\bar{b}$ satisfy* (2.1),
2°. *there exists $\varphi \in L^1$ with property* (1.4) *such that T_φ is invertible in A^p and the symbol $\varphi - ab$ satisfies* (1.7), *and*
3°. *the multipliers M_b and $M_{\bar{b}}$ are bounded from below as operators $A^p \to L^p$.*
Then, the operator T_a is Fredholm.

Proof. In order to show that $T_b T_\varphi^{-1}$ is a right parametrix for T_a we write

$$(2.17) \qquad T_a T_b = T_\varphi - T_{\varphi-ab} - PM_a H_b.$$

Here, the Toeplitz operator $T_{\varphi-ab}$ is compact by Theorem 1.3. By Lemma 2.1, the Hankel operator $H_b : A^p \to L^p$ is compact. To prove that $PM_a H_b$ is compact we need to show that PM_a is bounded in $H_b(A^p)$; this does not automatically follow from the boundedness of T_a since the domain of the operator is different from A^p. Now, for all $g \in M_b(A^p)$ we take $f \in A^p$ such that $g = bf$ and write

$$\|PM_a g\|_p = \|PM_{ab} f\|_p = \|PM_\varphi f - PM_{\varphi-ab} f\|_p$$
$$(2.18) \qquad \leq \|T_\varphi f\|_p + \|T_{\varphi-ab} f\|_p \leq C\|f\|_p,$$

by the assumption 2° and Theorem 1.3. Since M_b is bounded from below by the assumption 3°, we have $C\|f\|_p \leq C'\|g\|_p$ so that in view of (2.18), $PM_a : H_b(A^p) \to A^p$ is bounded. Thus $PM_a H_b$ is a compact operator and $T_b T_\varphi^{-1}$ is a right parametrix, since

$$(2.19) \qquad T_a T_b T_\varphi^{-1} = I - T_{\varphi-ab} T_\varphi^{-1} - PM_a H_b T_\varphi^{-1}.$$

where the last two terms are compact operators by what was proven above.

As for the left parametrix, we recall that the adjoint $T_a^* : A^q \to A^q$ of T_a coincides with $T_{\bar{a}}$, see Theorem 1.2 and [6], Proposition 7.1. By the assumptions of our proposition and the proof above, we find that

$$(2.20) \qquad T_{\bar\varphi} - T_{\bar a} T_{\bar b} : A^q \to A^q$$

is a compact operator, where $1/p + 1/q = 1$. (Notice that a function satisfies condition (1.7) if and only if its complex conjugate does.) Hence, by Schauder's theorem, also the operator

$$(2.21) \qquad \left(T_{\bar\varphi} - T_{\bar a} T_{\bar b}\right)^* = T_{\bar\varphi}^* - T_{\bar b}^* T_{\bar a}^* = T_\varphi - T_b T_a : A^p \to A^p$$

is compact so that $T_\varphi^{-1} T_b$ is a left parametrix. Thus, T_a is Fredholm. $\square$

3. EXAMPLES.

Let us consider some examples Fredholm Toeplitz operators T_a. Since the Fredholm property is connected with the compactness of Hankel operators and the latter is related with symbols belonging to the space VMO_δ, it is of interest to consider bounded symbols $a \notin VMO_\delta$. We start by constructing concrete examples of such symbols. Let us set

$$(3.1) \qquad \Theta(z) = \text{sgn}\big(\sin(1/(1-r))\big),$$

where $z = re^{i\theta} \in \mathbb{D}$ and we denote for all real numbers s,

$$(3.2) \qquad \text{sgn}(s) = \begin{cases} 1, & \text{if } s \geq 0 \\ -1, & \text{if } s < 0. \end{cases}$$

The proof in Remark 2.4 of [5] shows that Θ satisfies property (1.7). We present the details of this proof, since modifications of it will soon be needed. Let $D = D(1 - 2\delta, \theta)$ be as in (1.3) for some $\theta \in [0, 2\pi)$ and small enough $\delta > 0$ and consider $\zeta = \rho e^{i\phi} \in D$. Using the change of variable $y = 1/(1 - \varrho)$ we get

$$(3.3)\; 2|D||\hat{\Theta}_D(\zeta)| = \int\limits_{\theta}^{\phi} \Big| \int\limits_{1-2\delta}^{\rho} \mathrm{sgn}\big(\sin(1/(1-\varrho))\big)d\varrho \Big| \leq \pi\delta \Big| \int\limits_{1/(2\delta)}^{1/(1-\rho)} \frac{1}{y^2}\mathrm{sgn}(\sin y)\, dy \Big|$$

We divide the integration interval to the subintervals $J_n := [2\pi n, 2\pi(n+1)]$, $n \in \mathbb{N}$. We have

$$\Big| \int\limits_{J_n} \frac{1}{y^2}\mathrm{sgn}(\sin y)dy \Big| = \Big| \int\limits_{2\pi n}^{2\pi n + \pi} \Big(\frac{1}{y^2} - \frac{1}{(y+\pi)^2} \Big) dy \Big| \leq C \int\limits_{2\pi n}^{2\pi n + \pi} \frac{1}{y^3} dy \leq C' \frac{1}{n^3}$$

which yields

$$(3.4) \qquad \Big| \int\limits_{2\delta}^{1/(1-\rho)} \frac{1}{y^2}\mathrm{sgn}(\sin y)\, dy \Big| \leq C \sum_{n=[1/(2\delta)]}^{\infty} \frac{1}{n^3} \leq C'\delta^2,$$

where $[s]$ denotes the largest integer not bigger than $s \in \mathbb{R}$. We obtain (1.7) by taking into account (3.3) and that $|D|$ is of order δ^2.

We show that $\Theta \notin VMO_\delta$. Indeed, fixing $r > 0$ a calculation similar to (3.3)–(3.4) shows that the average $\hat{\Theta}_r$ of Θ over a set $B(z, r)$ has the bound $|\hat{\Theta}_r| \leq C\delta$, where $\delta = 1 - |z|$ for all $z \in \mathbb{D}$. On the other hand, we have $|\Theta(z)| = 1$ for all $z \in \mathbb{D}$ by the definition, hence, the quantity (1.8) has, for all z close enough to the unit circle, the lower bound

$$MO_r^p(\Theta)(z) \geq \Big(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} (|\Theta(\zeta)| - |\hat{\Theta}_r|)^p dA(\zeta) \Big)^{1/p}$$

$$\geq C' \Big(\frac{1}{|B(z,r)|} \int\limits_{B(z,r)} (1 - C\delta)^p dA(\zeta) \Big)^{1/p} \geq C''.$$

Thus, Θ belongs to BMO_δ but not to VMO_δ.

a) Consider the Toeplitz operator T_a with the bounded symbol

$$a = 1 + \Theta.$$

By the argument above and Theorem 1.3, T_Θ is compact in any space A^p so that T_a is Fredholm although $a \notin VMO_\delta$. Moreover, given any neighborhood $U \subset \overline{\mathbb{D}}$ of the boundary $\partial\mathbb{D}$, the symbol a vanishes in a subset of U with positive area.

More generally, assume that a function $f \in VMO_\delta$ satisfies (1.4) and in addition $|\hat{f}_D(\zeta)| \geq C > 0$ for all $D \in \mathcal{D}$ and $\zeta \in D$. (Trivial example: f is continuous, real and bounded from below by a positive constant on the closed unit disc.) The operator T_a with

$$a = f + \Theta$$

is Fredholm, although $a \notin VMO_\delta$. See Theorem 2.8 of [5]. The index of T_a is of course the same as that of T_f.

b) Obviously, the example a) can be modified by endlessly many ways. One could make Θ a little more slowly oscillating by replacing in (3.1) the function $1/(1-r)$ by $(1-r)^{-\alpha}$ for any constant $0 < \alpha < 1$. It could also be made unbounded by multiplying Θ with $(1-r)^{-\beta}$, for any $0 < \beta < 1$. Also, the symbol Θ of (3.1) is not smooth, but it can be written as the sum

$$(3.5) \qquad\qquad \Theta = \Theta_1 + \Theta_2$$

where Θ_1 is an infinitely smooth bounded function, a mollification of Θ, and Θ_2 induces a compact Toeplitz operator. Indeed, we set

$$(3.6) \qquad\qquad \Theta_1(z) = \int_{\mathbb{C}} J_z(w-z)\Theta(w)dA(w),$$

where J_z, $z \in \mathbb{D}$, is for example the scaled mollifier

$$J_z(w) = \nu(z)^2 J\big(\nu(z)w\big) \quad \text{and} \quad \nu(z) = e^{1/(1-|z|^2)} > 1$$

and J is the standard C^∞-mollifier

$$J(w) = \begin{cases} Ce^{-1/(1-|w|^2)}, & w \in \mathbb{D}, \\ 0, & w \in \mathbb{C}, \ |w| \geq 1 \end{cases}$$

with a normalization factor $C > 0$ such that $\int_{\mathbb{C}} J dA = 1$. We have $|\Theta_2(z)| \leq 1$ for all z. The support of Θ_2 is contained in the set

$$E = \big\{ z = re^{i\theta} \in \mathbb{D} : |r - r_n| \leq 1/\nu(r_n) = e^{-\pi n} \text{ for some } n \in \mathbb{N} \big\}$$

where $r_n = 1-1/(\pi n)$, $n \in \mathbb{N}$, are the discontinuity points of the function $\operatorname{sgn}(\sin(1/(1-r)))$, see (3.1). Thus, for any $D \in \mathcal{D}$, the area of a set $E \cap D$ is "small" in comparison with $|D|$, and one can use this to show that Θ_2 satisfies (1.7). Hence, T_{Θ_2} and $T_{\Theta_1} = T_{\Theta} - T_{\Theta_2}$ are compact and

$$T_{1-\Theta_1} \quad \text{and} \quad T_{f-\Theta_1},$$

with f as in the example a), are Fredholm; also, $\Theta_1 \notin VMO_\delta$. We leave it to the reader to verify the details of these claims.

The next example is the most relevant as regards to Theorem 2.2.

c) Let the symbol Θ with property (1.7) be as above, and let $f \in C^1(\mathbb{D})$ be a real function satisfying (2.1) such that

$$(3.7) \qquad\qquad C \leq f(z) \leq C_1$$

for some constants $C_1 > C > 0$ and all $z \in \mathbb{D}$. In view of Theorem 2.2, the symbol

$$(3.8) \qquad\qquad a = 1 - f\Theta$$

induces a Fredholm Toeplitz operator, since for $b = 1/f$ we have $ab = 1/f - \Theta$ and all conditions of the proposition are satisfied; notice that $\bar{\partial} b = (\bar{\partial} f)/f^2$.

Apparently, more general functions f and Θ could be used here, once Θ has property (1.7) and the function $b = 1/f$ satisfies 1° and 3° of Theorem 2.2 and the operator $T_{1/f}$ is invertible. We do not know if $f\Theta$ has property (1.7) for all such f and Θ. We refer to Proposition 2.2 of [19], where the product of a function with property (1.4) and another function has been studied.

d) Let us finally consider the symbol

$$(3.9) \qquad a = \frac{1}{i + \Theta/2},$$

where Θ is as in (3.1). We indicate two ways to show that $T_a : A^p \to A^p$ is Fredholm for all p, $1 < p < \infty$. First, let us define $b(z) = 5i/4$ so that $1°$ of Theorem 2.2 trivially holds. Moreover, we have $\Theta(z)^2 = 1/4$ and thus

$$\sum_{n=0}^{\infty} \tilde{a}(z)^{4n} = \frac{16}{15}$$

for all $z \in \mathbb{D}$, hence

$$a(z)b(z) = \frac{5}{4} \frac{1}{1 - i\Theta} = \frac{5}{4} \sum_{n=0}^{\infty} (i\Theta)^n = \frac{5}{4} \Big(\sum_{n=0}^{\infty} \tilde{a}^{4n} + i\Theta^{4n+1} - \Theta^{4n+2} - i\Theta^{4n+3} \Big)$$

$$(3.10) \qquad = \frac{5}{4} \Big(\frac{16}{15} - \frac{16}{4 \cdot 15} \Big) + \frac{5i}{4} \Theta \Big(\frac{16}{2 \cdot 15} - \frac{16}{8 \cdot 15} \Big) = 1 + \Theta \frac{i}{2}.$$

As conclusion, property $2°$ holds, by what was remarked on the symbol Θ. As for $3°$, this trivially holds for M_b and $M_{\bar{b}}$.

On the other hand one can also use a calculation similar to (3.10) to show directly that the symbol a can be written as $\alpha + \beta\Theta$ for some constants $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ so that the Fredholm property follows in the same way as in the example a).

4. Conclusions.

In the main general Theorem 2.2 we have combined an existing "weak" condition on the compactness of Toeplitz operators with an improved result on the compactness of Hankel operators, to get a new, weak condition for the Fredholm property of a given Toeplitz operator T_a. According to the basic intuition, for a very regular a, like a continuous one on the closed unit disc $\overline{\mathbb{D}}$, the symbol must be non-zero on the boundary, in order to induce a Fredholm Toeplitz operator. Passing to more general symbols a, it is a consequence of our main theorem that there are cases where, for any neighborhood U of the boundary $\partial\mathbb{D}$, a may vanish in a subset of U with positive measure. This counterintuitive result is explained by the cancellation phenomena, which according to the proof of the main theorem and the reference [5] may lead to compact Toeplitz operators.

Our theorem also leads to many examples of Fredholm Toeplitz operators, where the assumption on the VMO_δ-property of the symbol can be relaxed, although this condition often appears in the existing literature.

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