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Abstract

We study forward curves formed from commodity futures prices listed on the Standard and Poor's-Goldman Sachs Commodities Index (S&P GSCI) using recently developed tools in functional time series analysis. Functional tests for stationarity and serial correlation suggest that log-differenced forward curves may be generally considered as stationary and conditionally heteroscedastic sequences of functions. Several functional methods for forecasting forward curves that more accurately reflect the time to expiry of contracts are developed, and we found that these typically outperformed their multivariate counterparts, with the best among them using the method of predictive factors introduced by Kargin & Onatski (2008).

Keywords: Forward curves, S&P GSCI, Commodity Futures, Functional Data Analysis, Functional Autoregressive Models, Functional Principal Component Analysis

JEL: C12, C32, C58, G15, G17, Q02

1. Introduction

Commodity futures are one of the most widely traded types of financial assets. One reason for this is that they provide risk diversification for portfolio management, since the returns on commodity futures are negatively correlated with the return on stocks and bonds (Gorton & Rouwenhorst, 2006; Bhardwaj et al., 2015). The forward curve, also referred to as the futures term structure, is formed by the futures/forward prices for a particular commodity over all available maturities at a certain point in time. Empirically, commodity futures behave quite differently when compared to other conventional financial assets, such as stocks and bonds. Their stylized properties include normal backwardation (Litzenberger & Rabinowitz, 1995), mean reversion (Bessembinder et al., 1995; Kocagil et al., 2001), strong heteroscedasticity (Duffie et al., 1999), positive correlation between price volatility and the degree of backwardation (Ng & Pirrong, 1994; Litzenberger & Rabinowitz, 1995), the "Samuelson effect" (Samuelson, 1965), and pronounced seasonality (Sørensen, 2002). Each of these terms generally describe the shape and evolution of forward curves over time. See also (Routledge et al., 2000) for a summary of these findings.

Forward curves derived from commodity futures are important for both academic researchers and financial practitioners. From the academic perspective, the information contained in the forward curves (such as shape, slope, roll-yield) are essential input factors in pricing models of commodities (Schwartz & Smith, 2000; Pilipovic, 2007; Chong et al., 2017) and predictive factors for futures returns (Gorton et al., 2013). They can also be related to volatility of spot and futures prices (Haugom & Ullrich, 2012; Kogan et al., 2009). For practitioners forward curves are one of the key factors in investment decisions on commodities. A number of commodity trading strategies are built, fully or partially, on the forward curves (Mou, 2010; Fuertes et al., 2015; Gomes, 2015). Mou (2010) designed trading strategies to take advantage of the resulting impact on forward curves from the rolling activity conducted by commodity index funds on the Standard and Poor's-Goldman Sachs Commodity Index (S&P GSCI).

Without a doubt these methods could be improved with useful forecasts of full forward curves, although, to the best of our knowledge, little attention has been paid to this issue in the literature. The problem of forecasting forward curves is similar to that of forecasting the term structure of interest rates, which enjoys a comparably abundant and expanding literature (see Duffee et al., 2012; Diebold & Rudebusch, 2013, for a review). Among those methods, one of the most popular was developed by Diebold & Li (2006), who proposed a dynamic Nelson-Siegel model

(DNS) to forecast the yield curve as a three-dimensional parameter vector evolving dynamically. The three time-varying parameters in their model correspond to level, slope, and curvature. Many extensions of the DNS model have been proposed. Xiang & Zhu (2013) developed a DNS term structure model subject to regime shifts, and Byrne et al. (2017) extended the DNS model to allow the model dimension and the parameters to change over time. Their version of DNS considers a large set of macro-financial factors to characterize the nonlinear dynamics of yield factors.

The available literature on forecasting futures prices focuses primarily on the active contract[1] or front-month contract price (e.g. Moskowitz et al., 2012). Chantziara & Skiadopoulos (2008) employed principal component analysis (PCA) to investigate whether the daily evolution of the forward curves of petroleum futures (NYMEX crude oil, heating oil, gaosline, and IPE crude oil futures) can be forecasted, and found that the retained principal components have small forecasting power measured both within a training sample and out-of-sample. Baruník & Malinska (2016) proposed to forecast the forward curves of oil futures by coupling dynamic neural networks with the Nelson–Siegel model. Their forecasting strategy outperformed other benchmark models (vector autoregressive and random walk models) for crude oil futures prices. Grønborg & Lunde (2016) applied DNS model for the forward curves of oil futures contracts and obtained forecasts of prices of these contracts. Evaluated by the criteria of model confidence sets (Hansen et al., 2011), their models produce better forecasting results than conventional benchmarks (autoregressive and VAR). Power et al. (2017) used wavelet thresholding to de-noise futures price data before estimating a state-space model. For their sample of CBOT corn futures, de-noising the data by wavelet thresholding improved forecasting results in most cases. These forecasting methods build upon a large literature devoted to factor models for commodity futures (see Gibson & Schwartz, 1990; Litterman & Scheinkman, 1991; Schwartz & Smith, 2000; Tolmasky & Hindanov, 2002).

An important distinction in each of the above models is that they intrinsically treat the futures prices and forward curves as multivariate objects. This is natural to do since, for example, the futures prices of a commodity with contracts expiring each month in a twelve month period can be considered as a twelve dimensional vector. Doing so does, however, come with some information loss, since the components of such a vector will correspond to only approximate

[1] The active contract for a particular commodity is normally considered to be the contract with highest trading volume among all available maturities.

information on the time to expiry of the available contracts.

A less explored framework which offers new insight into this data is to consider them as sparse observations from an underlying continuous futures price curve. To illustrate, for a given commodity on day i with J available contracts, futures price information can be represented by $P_i(t_{i,j})$, where the $t_{i,j}$, $j \in \{1, 2, ..., J\}$, represent the time to expiration of the available contracts. Importantly, these times change each day and roll from one month to the next. This can easily be accounted for by interpolating the available contract prices in order to produce an estimate for the full forward curve, $P_i^{(C)}(t)$, and the series of these forward curves can be analyzed by functional time series analysis. The scope of methodology for analyzing such functional time series has increased tremendously since the seminal work of Bosq (2000) on the topic, see Hörmann & Kokoszka (2012) for a more recent review.

In this paper, we consider the analysis and predication of forward curves as functional data objects. In defining the forward curves as functional data, an important distinction is how one defines the time to expiry of a given contract. Defining this time in months gives rise to essentially equivalent methods as the available multivariate techniques referenced above, while using a higher resolution, like days or weeks, leads to quite different methods. It is shown below using recently developed tools in functional time series analysis that the log-differenced forward curves over several different temporal resolutions are reasonably stationary. We further show that for many commodities these curves can be thought of as weak functional white noises exhibiting conditional heteroscedasticity, suggesting that the forward curves evolve approximately as martingales. In several cases though the forward curves appear predictable in that they exhibit substantial autocorrelation. Given this, several models for forecasting forward curves are proposed and studied in a comprehensive data analysis. This analysis suggests that 1) measuring time to expiry of contracts in terms of days provides a significant improvement in terms of forecasting accuracy, and 2) the functional forecasting methods tend to outperform multivariate techniques, and in several cases are able to beat naïve forecasts in out-of-sample evaluations.

The rest of the paper is organized as follows. In Section 2, we discuss how we obtained and used the raw futures contract data in order to construct functional data objects at various temporal resolutions. Section 3 presents a functional time series analysis of the log-differenced forward curves, in particular presenting the results of stationarity and white noise tests to measure their suitability to be modelled with using stationary functional time series models. Methods for forecasting forward curves are put forward and compared in an extensive data analysis of commodity futures listed on the S&P GSCI in Section 4. Some concluding remarks and a summary of this work is given in Section 5, and some technical details about the stationarity and white noise tests as well as further results from the data analysis are presented in Appendix A.

Briefly we define some of the notation that we use below. We use $\mathbb{E}(\cdot)$ to denote mathematical expectation. We let $L^2[0,1]^d$ denote the space of real valued square integrable functions defined on unit hypercube $[0,1]^d$ of dimension d with norm $\|\cdot\|$ induced by the inner product $\langle x,y\rangle = \int_0^1 \cdots \int_0^1 x(t_1,...,t_d)y(t_1,...,t_d)dt_1 \dots dt_d$ for $x,y \in L^2[0,1]^d$, the dimension of the domain being clear based on the input function. Henceforth we write $\int$ instead of $\int_0^1$. We use $\xrightarrow{D}$ to denote convergence in distribution.

2. Forward curves as functional data objects

In this section, we detail how we obtained the data that we consider below, and how it was transformed into functional data objects.

2.1. Basic Information on Data

The widely tracked S&P GSCI is recognized as a leading measure of general price movements of commodity futures, and it is designed to be investable by including the most liquid commodity futures. There are 24 commodity futures in the current basket of S&P GSCI[2], and we considered each of them in our analysis. Table 1 gives information on these commodities, including the available contract months for each commodity. The expiry of included futures contracts is limited to a maximum of 1 year, as trading activity and liquidity decline sharply with increasing time to maturity (Dürr & Voegeli, 2009). We did not include spot prices in our analysis because the observable prices of some of the commodities considered do not exist, and further empirical studies have shown there are discrepancies between the real spot price and futures prices (Irwin et al., 2009).

The raw data that we consider are generic futures time series for these commodities, which we downloaded from a Bloomberg terminal[3], and consists of daily records (in most cases) from January 3rd, 2000 to August 31st, 2017[4]. Continuous generic futures time series are constructed

[2] `http://us.spindices.com/index-family/commodities/sp-gsci`

[3] The raw data was downloaded using the Bloomberg-R API of the R Package "Rblpapi" (Armstrong et al., 2017).

[4] The detailed data information is reported in Table 11 in Appendix C.

Table 1: Commodity Futures Information

Symbol	***Name***	***Exchange***	***Contract Month***											
			Jan	***Feb***	***Mar***	***Apr***	***May***	***Jun***	***Jul***	***Aug***	***Sep***	***Oct***	***Nov***	***Dec***
FC	Feeder Cattle	CME	✓		✓	✓	✓			✓	✓	✓	✓	
LC	Live Cattle	CME		✓		✓		✓		✓		✓		✓
LH	Lean Hogs	CME		✓		✓		✓	✓	✓		✓		✓
C	Corn	CBOT			✓		✓		✓		✓			✓
S	Soybean	CBOT	✓		✓		✓		✓	✓	✓		✓	
W	Wheat	CBOT			✓		✓		✓		✓			✓
CC	Cocoa	ICE			✓		✓		✓		✓			✓
KC	Coffee	ICE			✓		✓		✓		✓			✓
KW	Kansas Wheat	KBOT			✓		✓		✓		✓			✓
CT	Cotton	ICE			✓		✓		✓			✓		✓
SB	Sugar	ICE			✓		✓		✓			✓		
LA	Aluminum	LME	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
LL	Lead	LME	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
LP	Copper	LME	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
GC	Gold	COMEX		✓		✓		✓		✓		✓		✓
LN	Nickel	LME	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
SI	Silver	COMEX	✓		✓		✓		✓		✓			✓
LX	Zinc	LME	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
CL	Light Crude Oil	NYMEX	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
CO	Brent Crude Oil	ICE	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
HO	Heating Oil	NYMEX	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
PG	Gasoline	ICE	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
NG	Natural Gas	NYMEX	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
XB	RBOB Gasoline	NYMEX	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

from actual futures prices in order of their relative expiration. For example, for a given commodity on day i, the price of the next (first) expiring contract can be denoted as $P_i(t_{i,1})$, where $t_{i,1}$ is the amount of time until the next contract expires. Similarly one may obtain prices of contracts expiring in $t_{i,j}$ time units, $P_i(t_{i,2}), ..., P_i(t_{i,J})$, when J contract months are available. For many commodities with contracts available in each month, $J = 12$. The curves constructed from these price data, which we denote $P_i^{(C)}(t)$, are commonly referred to as the forward curves, and are often displayed after linear interpolation of the available contracts assuming the time increments $t_{i,j}$ are measured in months until expiry.

The time points at which price information is available are regularly changing (Schwartz, 1997; Grønborg & Lunde, 2016; Jin, 2017). This is caused by two sources. Firstly, the maturity of all contracts in the full contract chain decreases daily. The second source is due to rolling. At the date the front month contract expires, there is a jump to the expiry date of the next contract. Due to the nature of the irregular sampling time points in forward curves, traditional multivariate analysis techniques that treat the available contract prices as a multivariate vector with a balanced design are somewhat ill posed, since the coordinates correspond to contracts expiring at variable dates in the future. In addition, the time to expiry of futures contracts could be measured at several different temporal resolutions. Classically, this time is measured in months, although it is more accurate to measure the time to expiration in weeks or days. Table 2 describes the contract chain for the Corn futures and their maturity in the three resolutions of months, weeks and days to expiry during July 10$^{\text{th}}$ 2017 and July 19$^{\text{th}}$ 2017. The front contract rolled from the contract expiring in July (*C N7*) to the contract expiring in September (*C U7*) on July 17$^{\text{th}}$, due to the expiration of *C N7*. Figure 1 shows the corresponding segments of the forward curves, which are obtained by linear interpolation from $P_i(t_{i,j})$ $j = 1, ..., 5$ with $t_{i,j}$ measured at a daily resolution. Below we consider daily, weekly, and monthly resolutions for the times to expiry $t_{i,j}$.

Given the raw prices $P_i(t_{i,j})$, $j = 1, ..., J$, and a choice of the time resolution for a commodity, one can complete this data to full forward curves $P_i^{(C)}(t)$ using a number of methods. It is most typical in functional data analysis to perform this step using interpolation or smoothing techniques. One key difference in the interpretation of using interpolation versus smoothing to produce curves in this setting is the following: interpolation techniques effectively assume that the available contract prices are observed without error, while smoothing methods might be deemed more appropriate when the observed prices are thought to be contaminated by error, in which case a roughness penalty might be employed to derive fitted curves that ideally smooth through the

Table 2: Actual available contracts for Corn futures from 10 days in 2017, as well as the maturities of each contract in three resolutions.

	Actual Contracts Ticker				
	Generic 1	**Generic 2**	**Generic 3**	**Generic 4**	**Generic 5**
2017-07-10	C N7	C U7	C Z7	C H8	C K8
2017-07-11	C N7	C U7	C Z7	C H8	C K8
2017-07-12	C N7	C U7	C Z7	C H8	C K8
2017-07-13	C N7	C U7	C Z7	C H8	C K8
2017-07-14	C N7	C U7	C Z7	C H8	C K8
2017-07-17	C U7	C Z7	C H8	C K8	C N8
2017-07-18	C U7	C Z7	C H8	C K8	C N8
2017-07-19	C U7	C Z7	C H8	C K8	C N8
	Resolution: Daily				
2017-07-10	4	66	157	247	308
2017-07-11	3	65	156	246	307
2017-07-12	2	64	155	245	306
2017-07-13	1	63	154	244	305
2017-07-14	0	62	153	243	304
2017-07-17	59	150	240	301	361
2017-07-18	58	149	239	300	360
2017-07-19	57	148	238	299	359
	Resolution: Weekly				
2017-07-10	0	9	22	35	44
2017-07-11	0	9	22	35	43
2017-07-12	0	9	22	35	43
2017-07-13	0	9	22	34	43
2017-07-14	0	8	21	34	43
2017-07-17	8	21	34	43	51
2017-07-18	8	21	34	42	51
2017-07-19	8	21	34	42	51
	Resolution: Monthly				
2017-07-10	0	2	5	8	10
2017-07-11	0	2	5	8	10
2017-07-12	0	2	5	8	10
2017-07-13	0	2	5	8	10
2017-07-14	0	2	5	8	10
2017-07-17	2	5	8	10	12
2017-07-18	2	5	8	10	12
2017-07-19	2	5	8	10	12

Note: The contract ticker denotes the specific commodity futures contract with the expiry month and year: 1$^{\text{st}}$ letter – C (Corn); 2$^{\text{nd}}$ letter – H (March), K (May), N (July), U (September) & Z (December); and last number – 7 (2017) & 8(2018). For example, *C N7* is the corn contract expired in July 2017. The termination of trading of Corn futures on CBOT is the business day prior to the 15$^{\text{th}}$ calendar day of the contract month. The last trade day of *C N7* is on 14$^{\text{th}}$ July 2017. We define the time to expiration in three resolutions as follows. Daily resolution: the number of days between the last trade day and the current date. Weekly resolution: the number of weeks between the last trade day and the current date. If it comes to the same week of the last trade day, the time to expiration in weeks is set to be zero. Monthly resolution: the number of months between the last trade day and the current date. If it comes to the same month of the last trade day, the time to expiration in months is set to be zero.

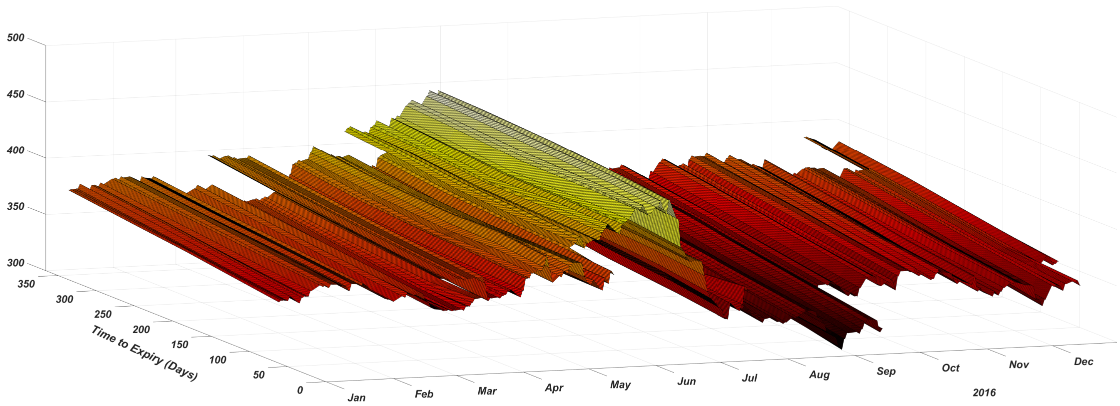


Figure 1: 252 forward curves of corn futures contracts in 2016 derived from available price information and linear interpolation.

errors. Furthermore, in such applications the choice of the smoothing or interpolation technique used should also reflect the primary goal of the analysis. For instance, it has been previously observed (see e.g. Grith et al., 2018) that under-smoothed representations of the sampled curves can be more effective when the goal is to estimate a principal component basis, although such representations may sacrifice accuracy in estimating each individual curve.

Although assuming the observed prices are errorless seems reasonable here, we have explored a number of methods following both paradigms to produce forward curves, including linear and B-spline interpolation and smoothing to different orders as in Kargin & Onatski (2008) to estimate forward curves. Ultimately we decided to produce the forward curves that we study using cubic Hermite polynomial interpolation (Fritsch & Carlson, 1980), which guarantees continuity of the first derivative. We found Hermite interpolation to be superior to B-spline interpolation and smoothing in this setting, both quantitatively in terms of forecasting error measures as presented in Section 4, and since smoothing splines visibly over-fit the forward curves for many commodities.

In what follows all forward curves that we consider are constructed in this way. Furthermore we assume that t in $P_i^{(C)}(t)$ is normalized to the unit interval, so that we may think of each forward curve as a function defined on $[0, 1]$. This is demonstrated in Figure 2 with comparing the forward curves obtained by cubic Hermite interpolation in three resolutions.

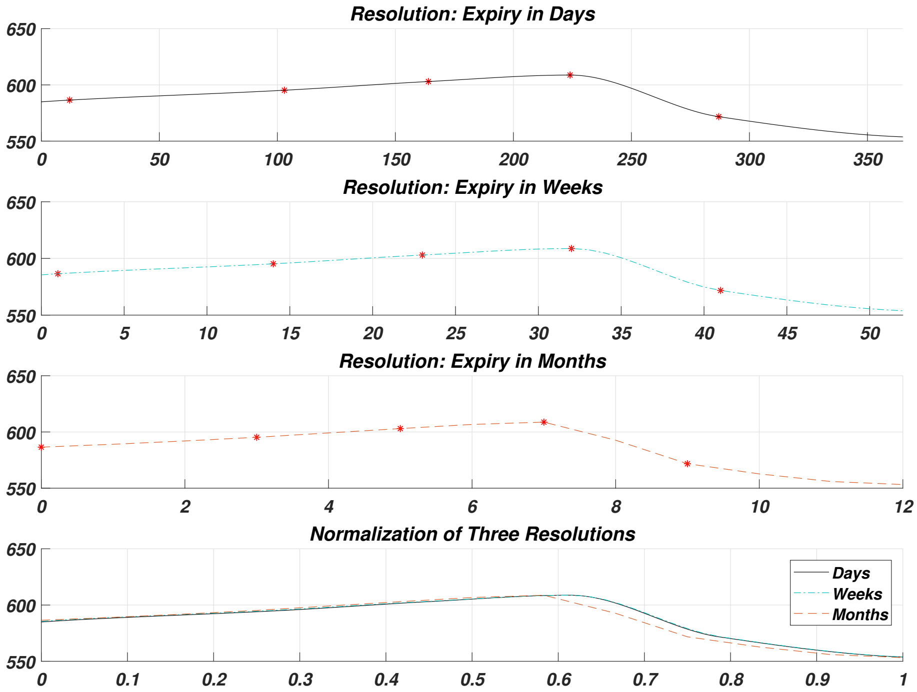


Figure 2: Comparison in three resolutions of $P_i^{(C)}(t)$ of corn futures contracts on December 2nd 2011. The curves are constructed by using cubic Hermite interpolation.

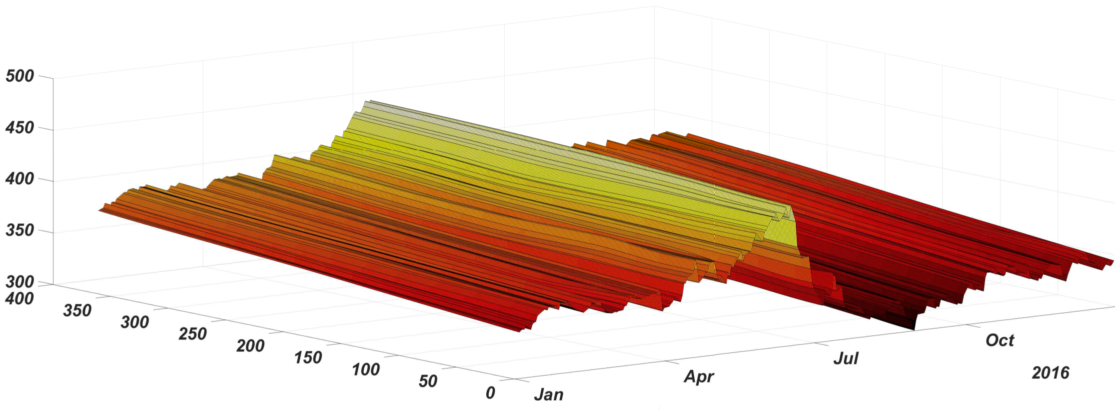
3. Functional time series analysis of forward curves

Our basic goal going forward is to evaluate the properties of the functional time series $P_i^{(C)}(t)$ defined from the commodity futures listed in the S&P GSCI, and measure to what extent they are predictable using methods to forecast functional time series. As is clear in the upper panel of Figure 3, the curves $P_i^{(C)}(t)$ are highly non-stationary, exhibiting strong seasonality and level shifts, and this encourages one to consider transformations of $P_i^{(C)}(t)$ that can be studied as stationary functional time series.

Definition 3.1. *Given a sequence of forward curves $P_i^{(C)}(t)$, $1 \leq i \leq N$ the log-differenced forward curves are defined by $X_1(t) = 0$, and*

$$X_i(t) = 100 \left[\log(P_i^{(C)}(t)) - \log(P_{i-1}^{(C)}(t)) \right], \quad 2 \leq i \leq N. \tag{1}$$

A plot of the log-differenced forward curves derived from Corn futures is given in the lower panel of Figure 3. In order to evaluate the stationarity of the log-differenced forward curves for



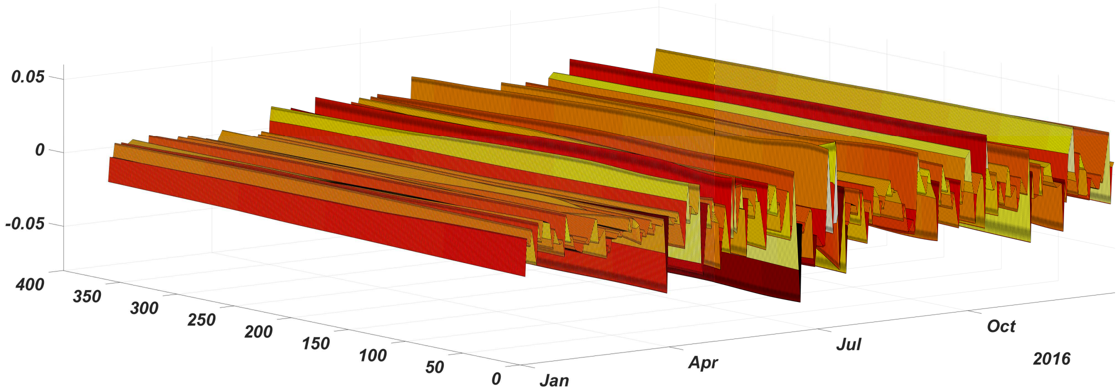


Figure 3: **Upper Panel**: 3D Plot of forward curves, $P_i^{(C)}(t)$, derived from corn futures contracts in 2016 using cubic Hermite interpolation. **Lower Panel**: 3D Plot of the log-differenced forward curves, $X_i(t)$, derived from corn futures contracts in 2016 using cubic Hermite interpolation. There are 252 curves for both panels in 2016.

each commodity listed in Table 1, we applied the KPSS test for functional time series developed in Horváth et al. (2014) to the entire sample $X_i(t)$, $1 \le i \le N$, which we now briefly describe. The functional analogue of the standard CUSUM process

$$Z_N(x,t) = \frac{1}{\sqrt{N}} \left[\sum_{i=1}^{\lfloor Nx \rfloor} X_i(t) - x \sum_{i=1}^{N} X_i(t) \right]$$

compares the sum of the first $\lfloor Nx \rfloor$ observations to the x fraction of the total sum.

The functional KPSS statistic is defined as

$$\text{fKPSS}_N = \iint Z_N^2(x,t)dtdx.$$

Under mild weak dependence and moment conditions on the log-differenced series $X_i(t)$, the statistic fKPSS_N satisfies that

$$\text{fKPSS}_N \xrightarrow{D} \Xi_{KPSS}, \ N \to \infty,$$

where Ξ_{KPSS} is a random variable whose distribution can be consistently estimated from the original curves $X_i(t)$; see Horváth et al. (2014). The specific settings we used to perform this estimation are detailed in Appendix A. Under several general departures from stationarity for the sequence $X_i(t)$, including change points in the mean function and unit-root type alternatives, fKPSS_N diverges in probability to positive infinity. A p-value for the null hypothesis of stationarity can then be obtained as $P(\Xi_{KPSS} > \text{fKPSS}_N)$. The p-values of these tests applied to the log-differenced forward curves at each temporal resolution are given in Table 3. The results suggest that it is reasonable to assume the log-differenced curves are stationary, which provides some justification for forecasting the time series $X_i(t)$ using stationary functional time series models. We consider such models and forecasts below in Section 4.

Assuming the series $X_i(t)$ is strictly stationary, we may define the autocovariance kernel of $X_i(t)$ at lag h by $c_h(t,s) = \text{cov}(X_0(t), X_h(s))$. c_h defines the lag h autocovariance operator by $C_h(f)(t) = \int c_h(t,s)f(s)ds = \mathbb{E}\left[\langle X_0, x\rangle X_h(t)\right]$, which is instrumental in estimating the functional autoregressive models that we consider below for the purpose of forecasting. If $c_h(t,s) = 0$ for all h larger than zero, then methods for forecasting $X_i(t)$ based on minimizing the mean squared forecast error will reduce to estimating $X_i(t)$ with its mean. It is hence worthwhile to test the hypothesis

$$\mathcal{H}_{0,K}: \quad c_h(t,s) = 0, \quad \text{for} \quad h = 1, ..., K.$$

Tests of $\mathcal{H}_{0,K}$ are typically referred to as portmanteau tests going back to the seminal work for scalar time series of Box & Pierce (1970). Recently a number of approaches have been proposed to test $\mathcal{H}_{0,K}$ with functional data; see Gabrys & Kokoszka (2007), Horváth et al. (2013), and Zhang (2016). In general these tests are built under stronger assumptions than $\mathcal{H}_{0,K}$ alone, such as

$$\mathcal{H}_{0,iid}: \quad X_i(t) \text{ are independent and identically distributed,}$$

or

$$\mathcal{H}_{0,ch}: \quad X_i(t) = \sigma_i(t)\varepsilon_i(t) \text{ where } \varepsilon_i(t) \text{ is a mean zero, independent}$$
$$\text{and identically distributed sequences of errors, and } \sigma_i^2(t) = g(\varepsilon_i, \varepsilon_{i-1}, ...).$$

A test of $\mathcal{H}_{0,ch}$ is meant to account for sequences that satisfy $\mathcal{H}_{0,K}$ as approximate martingale difference sequences exhibiting conditional heteroscedasticity, as with functional versions of generalised autoregressive conditionally heteroscedastic sequences. Such functional time series models have recently been proposed in Hörmann et al. (2013) and Aue et al. (2017). Evidently $\mathcal{H}_{0,K}$ holds under both $\mathcal{H}_{0,iid}$, and $\mathcal{H}_{0,ch}$, but these stronger assumptions allow for the construction of asymptotically validated tests. Here we apply the tests of $\mathcal{H}_{0,iid}$, and $\mathcal{H}_{0,ch}$ recently proposed by Kokoszka et al. (2017). In order to describe these tests, the autocovariance kernel c_h is estimated by

$$\hat{c}_h(t,s) = \frac{1}{N}\sum_{j=1}^{N-h}\{X_j(t) - \bar{X}(t)\}\{X_{j+h}(s) - \bar{X}(s)\}, \quad \bar{X}(t) = \frac{1}{N}\sum_{j=1}^{N} X_j(t),$$

and through these estimates we define a test statistic for $\mathcal{H}_{0,K}$ by

$$V_{N,K} = N\sum_{h=1}^{K}\|\hat{c}_h\|^2.$$

Under $\mathcal{H}_{0,iid}$ and some additional moment conditions, it follows that

$$V_{N,K} \xrightarrow{D} \Theta_{K,iid}, \quad N \to \infty,$$

where the distribution of $\Theta_{K,iid}$ can be consistently estimated from the sample. Similarly under $\mathcal{H}_{0,ch}$ and some further moment and weak dependence conditions, it follows that

$$V_{N,K} \xrightarrow{D} \Theta_{K,ch}, \quad N \to \infty,$$

and again the distribution of $\Theta_{K,ch}$ can be consistently estimated. Approximate p-values for $\mathcal{H}_{0,iid}$ and $\mathcal{H}_{0,ch}$ are then given by $p = P(\Theta_{K,iid} > V_{N,K})$ and $p = P(\Theta_{K,ch} > V_{N,K})$, respectively. These results and the necessary conditions under which they hold are detailed in Kokoszka et al. (2017), and the specific implementations of these tests that we use here are described further in Appendix A. The results of these tests applied to the log-differenced forward curves for each commodity with $K = 10$ are given in Table 3, which indicate that these curves display stronger autocovariance that is consistent with the sequence being independent and identically distributed. Most commodities reject $\mathcal{H}_{0,iid}$ but cannot reject $\mathcal{H}_{0,ch}$, suggesting that in general the log-differenced forward curves evolve as a dependent but uncorrelated sequence of functions. However, there remains a relatively large subset of commodities (FC, LC, LH, CL, CO, HO, and NG) that reject both $\mathcal{H}_{0,iid}$ and $\mathcal{H}_{0,ch}$, indicating strong autocovariance.

Table 3: Results of the functional KPSS test and portmanteau tests in terms of p-values when applied to the log-differenced forward curves for each commodity with temporal resolutions of days, weeks, and months.

Resolution:	**Days**			**Weeks**			**Months**		
Symbol/Test:	**fKPSS**	$\mathcal{H}_{0,iid}$	$\mathcal{H}_{0,ch}$	**fKPSS**	$\mathcal{H}_{0,iid}$	$\mathcal{H}_{0,ch}$	**fKPSS**	$\mathcal{H}_{0,iid}$	$\mathcal{H}_{0,ch}$
FC	68.3%	***0.0%***	***0.0%***	67.5%	***0.0%***	***0.0%***	99.3%	***0.0%***	***0.0%***
LC	77.9%	***0.0%***	***0.0%***	77.5%	***0.0%***	***0.1%***	100.0%	***0.0%***	***0.4%***
LH	100.0%	***0.0%***	***0.0%***	100.0%	***0.0%***	***0.0%***	100.0%	***0.0%***	***0.0%***
C	59.1%	***0.0%***	82.8%	62.9%	***0.0%***	86.0%	97.5%	***0.0%***	84.4%
S	62.4%	***0.0%***	76.5%	59.8%	***0.0%***	75.2%	99.4%	***0.0%***	72.6%
W	34.8%	***0.0%***	99.0%	33.6%	***0.0%***	99.5%	87.4%	***0.0%***	96.7%
CC	19.2%	***0.0%***	49.4%	20.5%	***0.0%***	57.7%	69.0%	***0.0%***	64.1%
KC	52.5%	***0.0%***	14.2%	55.7%	***0.0%***	11.8%	95.4%	***0.0%***	23.5%
KW	31.4%	***0.0%***	88.2%	32.8%	***0.0%***	92.9%	87.8%	***0.0%***	86.3%
CT	93.0%	***0.0%***	10.3%	93.6%	***0.0%***	9.9%	100.0%	***0.0%***	17.8%
SB	64.3%	***0.0%***	17.8%	64.6%	***0.0%***	18.7%	98.3%	***0.0%***	22.7%
LA	82.4%	***0.0%***	53.5%	80.3%	***0.0%***	56.4%	99.8%	***0.0%***	57.1%
LL	49.7%	***0.0%***	28.3%	45.8%	***0.0%***	38.0%	91.2%	***0.0%***	34.9%
LP	49.5%	***0.0%***	17.6%	49.8%	***0.0%***	21.5%	92.0%	***0.0%***	18.6%
GC	12.6%	***0.0%***	90.8%	12.1%	***0.0%***	88.7%	58.9%	***0.0%***	90.7%
LN	41.4%	***0.0%***	89.9%	38.5%	***0.0%***	87.6%	88.1%	***0.0%***	90.6%
SI	34.8%	***0.0%***	93.0%	33.2%	***0.0%***	90.5%	84.1%	***0.0%***	92.0%
LX	66.4%	***0.0%***	35.7%	66.4%	***0.0%***	34.1%	97.9%	***0.0%***	28.3%
CL	23.6%	***0.0%***	***1.5%***	21.9%	***0.0%***	***3.9%***	70.1%	***0.0%***	***4.8%***
CO	18.3%	***0.0%***	***0.3%***	19.6%	***0.0%***	***0.4%***	70.2%	***0.0%***	***0.5%***
HO	22.2%	***0.0%***	***0.8%***	21.7%	***0.0%***	***1.5%***	74.4%	***0.0%***	***1.2%***
PG	90.5%	***0.0%***	44.2%	90.8%	***0.0%***	42.4%	100.0%	59.5%	55.9%
NG	25.5%	***0.0%***	***2.7%***	24.4%	***0.0%***	***1.4%***	89.0%	***0.1%***	***2.7%***
XB	77.5%	***0.0%***	37.9%	78.5%	***0.0%***	34.6%	99.9%	26.4%	52.1%

Note: the parameter $K = 10$ in the portmanteau tests. Red bold values indicate that they are less than 5%.

4. A comparison of forecasting methods for forward curves

The fact that quite a large number of commodities have log-differenced forward curves exhibiting strong serial correlation suggests that they might be predictable using linear forecasting techniques for curves. In this section we develop several approaches for forecasting the full forward curves, and provide a comparative study of each method for one step (one-day) ahead forecasting. The basic theory of linear functional time series forecasting was put forward in Bosq (2000), with the most widely employed method being functional autoregression. The functional autoregressive model of order $\mathcal{K}$ takes the form

$$\text{FAR}(\mathcal{K}):\ X_n(t) = \Psi_1(X_{n-1})(t) + \cdots + \Psi_{\mathcal{K}}(X_{n-\mathcal{K}})(t) + \varepsilon_n(t), \tag{2}$$

where Ψ_i $1 \leq i \leq \mathcal{K}$ is a kernel integral operator with kernel $\psi_i(t,s)$, i.e. $\Psi_i(f)(t) = \int \psi_i(t,s)f(s)ds$. Given the evident seasonal patterns in forward curves (Routledge et al., 2000; Sørensen, 2002; Borovkova & Geman, 2006), it is also reasonable to consider functional autoregressive models including lags of the series $X_i(t)$ corresponding to the clear seasonal variations in the price, for instance one week, month, quarter, etc.. We define the functional seasonal autoregressive model as

$$\text{FSAR}:\ X_n(t) = \Psi_{\ell,1}(X_{n-\ell_1})(t) + \cdots + \Psi_{\ell,k}(X_{n-\ell_k})(t) + \varepsilon_n(t), \tag{3}$$

where the $\Psi_{\ell,i}$ $1 \leq i \leq k$ are kernel integral operators with kernels $\psi_{\ell,i}(t,s)$, and $\ell_1, ..., \ell_k$ are specified lags.

These models can be estimated in a number of ways, and we consider two distinct methods. First we present an FPCA based estimator of Ψ_1 in the FAR(1) model, and similar estimators can be defined for the FAR($\mathcal{K}$) and FSAR models, which we develop in Appendix A. Following the principle of least squares, a natural estimator of Ψ_1 would be given by $\hat{\Psi}_1 = \hat{C}_1\hat{C}_0^{-1}$, where the functional covariance C_0 and C_1 are the lag zero and one autocovariance operators of the sequence $X_i(t)$. The empirical covariance and empirical lag-1 autocovariance operator can be estimated by

$$\begin{aligned}
\hat{C}_0(x)(t) &= \frac{1}{N}\sum_{j=1}^{N}\langle X_j - \bar{X}_1, x\rangle \left[X_j(t) - \bar{X}(t)\right], \\
\hat{C}_1(x)(t) &= \frac{1}{N}\sum_{k=1}^{N-1}\langle X_k - \bar{X}_1, x\rangle \left[X_{k+1}(t) - \bar{X}(t)\right],
\end{aligned} \tag{4}$$

which are each kernel integral operators with kernel $\hat{c}_0(t,s)$ and $\hat{c}_1(t,s)$, respectively. As discussed in Horváth & Kokoszka (2012), the inverse of the covariance operator C_0 is not bounded, which can be handled by projecting C_0 onto the subspace generated by the p leading functional principal components. In the literature of functional data analysis, the choice of p has been considered extensively, but no theoretical results regarding an "optimal" choice for forecasting purposes have been established. Usually, p is chosen with the objective that the largest p principal components explain a specified, high percentage of the "variance" of the observations (see e.g., Ramsay & Silverman, 2002), such as 90%, 95% or 99%. When the main goal is forecasting, cross–validation may be used to select p in order to avoid overfitting (e.g., Kargin & Onatski, 2008). Denote the empirical principal components (PCs) of the functional observations $X_i(t)$ as, $\hat{\nu}_j$, $j = 1, 2, 3..., p$. The inverse operator C_0^{-1} using the first p PCs is defined as

$$\hat{\Gamma}_p(x)(t) = \sum_{j=1}^{p} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle \hat{\nu}_j(t),$$

from which we obtain the estimator for $\Psi_{1,p}$

$$\begin{aligned} \hat{\Psi}_{1,p}(x) &= \hat{C}_1 \hat{\Gamma}_p(x) \\ &= \hat{C}_1 \left(\sum_{j=1}^{p} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle \hat{\nu}_j \right) \\ &= \frac{1}{N-1} \sum_{k=1}^{N-1} \left\langle X_k, \sum_{j=1}^{p} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle \hat{\nu}_j \right\rangle X_{k+1} \\ &= \frac{1}{N-1} \sum_{k=1}^{N-1} \sum_{j=1}^{p} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle \langle X_k, \hat{\nu}_j \rangle X_{k+1}. \end{aligned}$$

If a smoothing procedure is applied to $X_{k+1}(t) \approx \sum_{i=1}^{p} \langle X_{k+1}, \hat{\nu}_i \rangle \hat{\nu}_i(t)$, the estimator further becomes

$$\hat{\Psi}_{1,p}(x) = \frac{1}{N-1} \sum_{k=1}^{N-1} \sum_{j=1}^{p} \sum_{i=1}^{p} \hat{\lambda}_j^{-1} \langle x, \hat{\nu}_j \rangle \langle X_k, \hat{\nu}_j \rangle \langle X_{k+1}, \hat{\nu}_i \rangle \hat{\nu}_i. \tag{5}$$

The estimated kernel for the operator in equation 5 is

$$\hat{\psi}_{1,p}(t,s) = \frac{1}{N-1} \sum_{k=1}^{N-1} \sum_{j=1}^{p} \sum_{i=1}^{p} \hat{\lambda}_j^{-1} \langle X_k, \hat{\nu}_j \rangle \langle X_{k+1}, \hat{\nu}_i \rangle \hat{\nu}_j(s) \hat{\nu}_i(t)$$

Using the estimated kernel, we can make 1-step predictions as

$$\begin{aligned} \hat{X}_{n+1}(t) &= \int \hat{\psi}_{1,p}(t,s) X_n(s) ds \\ &= \sum_{j=1}^{p} \left(\sum_{i=1}^{p} \hat{\psi}_{1,p} \langle X_n, \hat{\nu}_i \rangle \right) \hat{\nu}_j(t) \end{aligned} \tag{6}$$

An alternative approach to estimating the FAR(1) model is to use the method of predictive factors, as proposed in Kargin & Onatski (2008). Effectively, this method aims to perform the dimension reduction step in estimating Ψ_1 in such a way that it minimizes the one step prediction error. It can be described as follows. The method aims to find an operator A that minimizes the prediction error

$$\min_{A \in \mathcal{R}_p} \left\{ \mathbb{E} \| X_{n+1} - A(X_n) \|^2 \right\} \tag{7}$$

where R_p is the set of all rank p operators mapping $L^2[0,1]$ into a subspace of dimension p. It was shown in Kargin & Onatski (2008) that this minimization can be approximately achieved using an operator of the form

$$\Psi_{1,p}(y) \approx \sum_{i=1}^{p} \langle y, b_i \rangle \, C_1(b_i), \quad b_i = C_0^{-1/2}(x_i),$$

where the processes $\{\langle y, b_i \rangle, i = 1, ..., p)\}$ are termed the predictive factors, and the functions $\{C_1(b_i), i = 1, ..., p)\}$ are termed the corresponding predictive loadings, which are the most relevant "directions" for one step ahead mean squared norm prediction. Again since $C_0^{-1/2}$ is not bounded, Kargin & Onatski (2008) proposed the following regularized estimator. With

$$\hat{\Phi}_{\alpha,1} = \hat{C}_{0,\alpha}^{-1/2} \hat{C}_1^T \hat{C}_1 \hat{C}_{0,\alpha}^{-1/2}, \quad \hat{C}_{0,\alpha} = \hat{C}_0 + \alpha I,$$

the operator $\Psi_{1,p}$ is estimated by

$$\hat{\Psi}_{\alpha,1,p}(x)(t) = \sum_{i=1}^{p} \left\langle x, \hat{b}_{\alpha,i} \right\rangle \hat{C}_1(\hat{b}_{\alpha,i})(t), \quad \hat{b}_{\alpha,i} = \hat{C}_{0,\alpha}^{-1/2}(\hat{x}_{\alpha,i})$$

where $\alpha \in \mathbb{R}^+$ is a regularisation parameter, I is the identity operator, and $\hat{x}_{\alpha,1}, ..., \hat{x}_{\alpha,k}$ are the eigenfunctions of $\hat{\Phi}_{\alpha,1}$. Then the 1-step prediction is determined by

$$\hat{X}_{n+1}(t) = \hat{\Psi}_{\alpha,1,p}(X_n)(t). \tag{8}$$

We follow Kargin & Onatski (2008) to select the number of predictive factors and the regularisation parameter by cross validation. We also implemented the technique of predictive factors with additional lags added, and found that these models tended to perform similarly to the single lag model, but at a much higher computational cost due to applying cross validation to select optimized values for multiple predictive factor regularization parameters.

It is worthwhile at this point to describe the primary similarities and differences between the FPCA and predictive factors estimation procedures. Although the motivation for the predictive factors approach derives from finding a low rank linear operator to minimize the one step prediction error in (7), and hence might appear more suitable for the forecasting problem at hand, since the FAR(1) framework is considered in both cases they each ultimately aim to estimate the same operator $C_1 C_0^{-1}$. Fundamentally then the differences derive from how the operator C_0^{-1} is estimated. In the FPCA case it is essentially assumed that the observed data are contained in the linear span of a finite collection of principal components in order that C_0^{-1} is bounded. The predictive factors approach implements a Tychonoff regularization that is tuned with cross validation to estimate C_0^{-1}. As such, one may understand the difference in performance of these methods in terms of the relative performance in prediction problems of principal component analysis versus regularisation techniques. It has been argued empirically that regularisation techniques tend to perform better in such cases, see Hall & Horowitz (2007) for a discussion in the setting of functional data, and we see below that the predictive factors technique of Kargin & Onatski (2008) tends in general to outperform the FPCA based methods.

4.1. Application to forecasting forward curves

We now provide the results of a comparative study of the above forecasting methods applied to forecasting forward curves. Since the functional forecasting models are based on the log-differenced forward curves, we used a given period of data to fit each of the proposed functional forecasting models and then to forecast the log-differenced forward curve $\hat{X}_{n+1}(t)$ by (6) or (8), which from (1) leads to the forecasted curve at trading day $n+1$ as $\hat{P}^{(C)}_{n+1}(t) = \exp\{[\hat{X}_{n+1}(t) + 100\log(P^{(C)}_n(t))]/100\}$. The specific models[5] that we considered for the log-differenced forward curves are:

- **FAR(1)**: Functional autoregressive process of order one
- **FAR(2)**: Functional autoregressive process with lags $[1, 2]$.
- **FAR(4)**: Functional autoregressive process with lags $[1, 2, 3, 4]$.
- **FSAR**: Functional seasonal autoregressive with lags $[1, 5, 21, 63]$.
- **FAR(1)-PF**: Functional autoregressive process of order one estimated using predictive factors.

[5] FAR($\mathcal{K}$) has lags at $1, 2, ...\mathcal{K}$, while FSAR has selected lags at $\ell_1, ..., \ell_k$ due to seasonality.

We also compared these to forecasts $\hat{\mathbf{P}}_i$ of the multivariate vector $\mathbf{P}_i = (P_i(t_{i,1}), ..., P_i(t_{1,J}))^\top$ using the method of Diebold & Li (2006), as well as naïve forecasts for the purpose of comparison. The benchmarks we compared to then were:

- **DL**: The method of Diebold & Li (2006), which is based on the dynamic Nelson–Siegel model with three factors of level, slope and curvature. We used the MATLAB implementation given in the Financial Instruments Toolbox of The MathWorks, Inc. (2019) applied to $\mathbf{P}_i$.
- **naïve**: Use current forward curve to predict the forward curve in the next day, $\hat{P}_{n+1}^{(C)}(t) = P_n^{(C)}(t)$.

In order to measure the forecast error, we considered the following four measures:

$$\text{Functional Error: } Err_f = \left(\frac{1}{N} \sum_{i=1}^{N} \int_{t_{i,1}}^{t_{i,J}} \left(\hat{P}_i^{(C)}(t) - P_i^{(C)}(t) \right)^2 dt \right)^{1/2}$$

$$\text{Relative Functional Error: } RErr_f = \left(\frac{1}{N} \sum_{i=1}^{N} \frac{\int_{t_{i,1}}^{t_{i,J}} \left(\hat{P}_i^{(C)}(t) - P_i^{(C)}(t) \right)^2 dt}{\|P_i^{(C)}(t)\|^2} \right)^{1/2}$$

$$\text{Multivariate Error: } Err_m = \left(\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} \left(\hat{P}_i(t_{i,j}) - P_i(t_{i,j}) \right)^2 \right)^{1/2}$$

$$\text{Relative Multivariate Error: } RErr_m = \left(\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} \frac{\left(\hat{P}_i(t_{i,j}) - P_i(t_{i,j}) \right)^2}{P_i(t_{i,j})^2} \right)^{1/2}$$

where $P_i(t_{i,j})$ denotes the raw prices that $t_{i,j}$ are the time points at which price information is available, and $\hat{P}_i(t_{i,j})$ is identified by extracting the values at the certain time points $t_{i,j}$ from the forecasted functional curve $\hat{P}_i^{(C)}(t)$.

Before examining the performance of the above methods, an important consideration in each of the proposed forecasting procedures is the selection of various hyperparameters, including the number of the principal components in FAR and FSAR, the number of the predictive factors and the value of the regularization parameter α in PF, and the parameter λ in the method of Diebold & Li (2006). To find the optimized values of hyperparameters, we performed times series cross validation (TSCV); see for example Hart (1994) for a description. Specifically we split the whole data set $D_n(t) = \left\{ P_n(t), P_n^{(C)}(t), X_n(t) \right\}, n = 1, ..., N$ into three segments of equal length[6], which

[6] The specific start date and end date of three segments for all commodity futures are reported in Table 11.

we refer to as training data $\{D_1(t), ..., D_{\lfloor N/3 \rfloor}(t)\}$, validation data $\{D_{\lfloor N/3 \rfloor+1}(t), ..., D_{\lfloor 2N/3 \rfloor}(t)\}$, and test data $\{D_{\lfloor 2N/3 \rfloor+1}(t), ..., D_N(t)\}$. We use an expanding window starting with the training data to predict the data one step ahead in the validation set. The hyperparameters are selected to minimize the total error in predicting each curve in the validation set[7]. Selected number of the principal components and the predictive factors are typically ranging between 1 and 5. After obtaining the optimized hyperparameters, we are able to examine the forecasting performance in the test period. Similar to the TSCV procedure, we conducted subsequent forecasts using an expanding window scheme[8], i.e. the model was re-estimated with all data up to date n in order to produce the forecast for the next observation at date $n+1$. The forecast of nine randomly selected Corn forward curves in the test period are displayed in Figure 4.

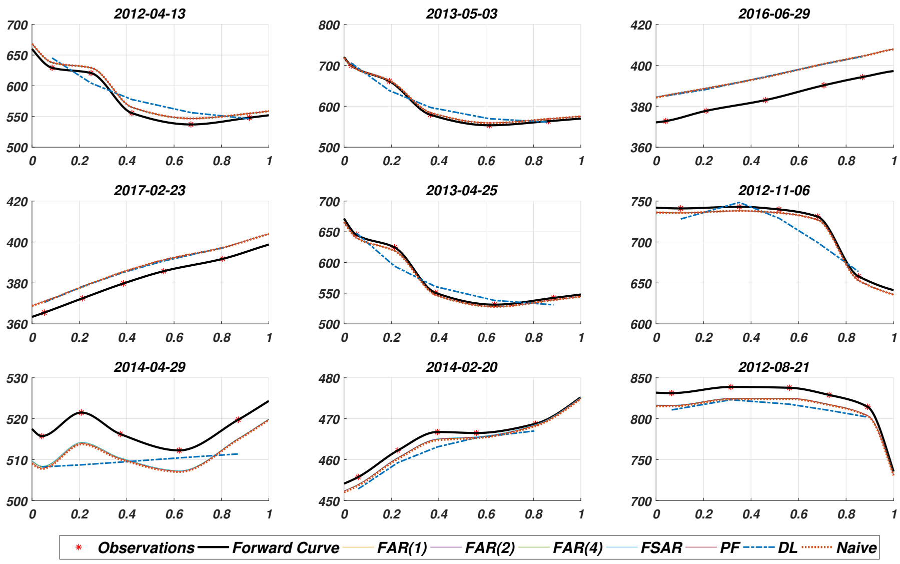


Figure 4: Forecasts from different models using daily resolution corn futures forward curves from nine randomly selected dates.

[7]The error measure for a one–step–ahead predictor with a given set of hyperparameters is calculated in the following way: 1) fit the model to the data $\{D_1(t), \cdots, D_n(t)\}$ and make a one–step–ahead prediction for the forward curve; 2) repeat Step 1 for $n = \lfloor N/3 \rfloor + 1, ..., \lfloor 2N/3 \rfloor$, where $\lfloor N/3 \rfloor + 1$ and $\lfloor 2N/3 \rfloor$ are the start date and the end date in the validation period, respectively; 3) compute the error measure over the validation period.

[8]To be specific, the forecasting for the test period is conducted by the following procedure: i) using the optimized value of hyperparameters, fit the model to the data $\{D_1(t), ..., D_n(t)\}$ and make a one–step–ahead prediction for the forward curve; ii) repeat Step i for $n = \lfloor 2N/3 \rfloor + 1, ..., N$, where $\lfloor 2N/3 \rfloor + 1$ and N are the start date and the end date in the test period, respectively; iii) Compute the error measure over the test period.

We evaluated each of these measures taken both over the validation period as well as over the test data to measure their predictive power. First, we considered how the choice of the temporal resolution of the forward curves affected forecasts. Table 4 shows the average relative forecasting error over all commodities for each model considered for the daily, weekly, and monthly resolution curves. From this it is clear that considering the daily resolution substantially improves forecasting for the functional methods. By the measure of relative multivariate error on the average of all commodities, the functional forecasting methods outperform the multivariate method of DL in all three resolutions.

The prediction errors for each model and across commodities in both validation and test periods are given in Tables 5 and 6 when the forward curves are constructed at the daily resolution; similar tables at the weekly and monthly resolution are given in Appendix B. Across all commodities the functional methods tended to significantly outperform the multivariate forecasting approach of Diebold & Li (2006). When comparing error rates in the validation period, PF produces the best forecasting performance for most commodities according to the multivariate error. Naïve method also have smallest functional error measure for several commodities. FAR methods can occasionally outperform the PF and the naïve methods. In terms of prediction error in the testing period, in many cases the naïve forecast was superior to the other forecasting methods considered, which agrees with the results of the portmanteau tests in Table 3. According to our test results, most of those forward curves evolve as dependent but uncorrelated sequences of functions. However, for some commodities (FC, LH, CL, and NG) the method of predictive factors was able to beat the naïve forecast in the test period regardless of which error measure is used. Some of these commodities interestingly coincides with those commodities whose log-differenced forward curves were measured to be predictable by the portmanteau tests. In these cases the multivariate approach of Diebold & Li (2006) was still worse than the naïve forecast in the relative forecasting performance.

The relative superiority of the PF method compared to the principal component based methods were to some extent expected. In the multivariate setting, regularized estimators are known to have good out-of-sample forecasting performance relative to least square estimators (e.g. Bernardini & Cubadda, 2015). Also, as is investigated and discussed in Bai & Ng (2019), such regularized principal component estimators are more robust when compared to standard PCA to large model errors, which may be thought to arise in this application from "spikes" in the curves caused by expiring contracts and the rollover effect. The performance of the multivariate procedure of Diebold & Li (2006) we believe can be explained by its reliance on the three factors

Table 4: Average relative forecasting error across commodities for daily, weekly, and monthly resolution forward curves.

Error Measure	Model\Resolution	Validation Period day	Validation Period week	Validation Period month	Test Period day	Test Period week	Test Period month
Relative Functional Error	**FAR(1)**	***1.9039%***	1.9079%	1.9410%	***1.4484%***	1.4540%	1.4916%
	FAR(2)	***1.9044%***	1.9085%	1.9415%	***1.4486%***	1.4543%	1.4921%
	FAR(4)	***1.9066%***	1.9106%	1.9433%	***1.4495%***	1.4552%	1.4929%
	FSAR	***1.9057%***	1.9097%	1.9426%	***1.4502%***	1.4557%	1.4929%
	PF	***1.9007%***	1.9049%	1.9384%	***1.4436%***	1.4493%	1.4877%
	Naïve	***1.9005%***	1.9045%	1.9376%	***1.4433%***	1.4489%	1.4871%
Relative Multivariate Error	**FAR(1)**	***5.4790%***	5.4939%	5.5973%	***4.1778%***	4.1944%	4.3203%
	FAR(2)	***5.4805%***	5.4959%	5.5985%	***4.1783%***	4.1950%	4.3216%
	FAR(4)	***5.4868%***	5.5018%	5.6041%	***4.1800%***	4.1970%	4.3235%
	FSAR	***5.4841%***	5.4989%	5.6020%	***4.1837%***	4.1998%	4.3245%
	PF	***5.4699%***	5.4852%	5.5893%	***4.1622%***	4.1790%	4.3078%
	Naïve	***5.4685%***	5.4836%	5.5870%	***4.1618%***	4.1784%	4.3067%
	DL		***7.2428%***			***5.6860%***	

Note: red bold values indicate that they are smallest in the three resolutions. The multivariate method of DL has the same performance for the three resolutions.

of level, slope, and curvature. Although the three factors could explain well for the term structure of government bond yields, it appears to not be sufficient for the more complicated shape of forward curves derived from commodity futures. By comparison, the principal components and predictive factors appear to provide more flexibility in describing the variety of underlying shapes of the forward curve over time.

5. Conclusion

We provide a thorough analysis using new tools in functional time series analysis of forward curves derived from a number of commodity futures listed in the S&P GSCI. We considered how one can interpolate the raw contract information to form full forward curves using cubic Hermite polynomials, and studied how constructing the curves using different temporal units to measure the time to expiry of the contract affects subsequent analysis of the curves. Regardless of the resolution used to measure the time to expiry, forward curves themselves are apparently non-stationary, however the log-differenced forward curves do appear to be stationary as measured by a functional analog of the KPSS test. Portmanteau tests indicate that for most commodities the amount of autocovariance observed in the sequence of curves is more than one would expect for a strong white noise sequence of curves, but is in accordance with what might be observed from

Table 5: Forecasting performance evaluated using functional and multivariate error measures Err_f and Err_m for commodity futures forward curves at a daily resolution.

Validation Period													
Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	0.795	0.794	0.794	0.795	**0.788**	0.793	2.324	2.324	2.322	2.325	**2.307**	2.612	2.321
LC	0.730	0.730	0.731	0.731	**0.730**	0.730	1.887	1.888	1.890	1.890	**1.886**	3.339	1.886
LH	0.819	0.819	0.819	0.818	**0.816**	0.819	2.305	2.304	2.305	2.303	**2.302**	5.362	2.305
C	9.231	9.226	9.232	9.243	**9.214**	9.217	22.000	21.993	22.007	22.033	**21.958**	25.480	21.968
S	17.228	**17.218**	17.242	17.265	17.223	17.219	47.640	47.613	47.679	47.733	47.622	53.755	**47.612**
W	15.161	15.139	15.147	15.155	15.131	**15.130**	36.009	35.958	35.985	36.002	**35.932**	42.133	35.937
CC	44.948	44.970	44.998	45.049	44.961	**44.945**	**106.597**	106.671	106.749	106.862	106.640	106.966	106.600
KC	**2.754**	2.759	2.761	2.757	2.758	2.758	**6.513**	6.526	6.531	6.521	6.523	6.605	6.523
KW	14.682	14.679	14.680	14.653	**14.596**	14.620	34.814	34.813	34.819	34.745	**34.623**	40.150	34.681
CT	1.611	1.611	**1.608**	1.612	1.609	1.610	3.844	3.844	**3.836**	3.846	3.838	4.616	3.841
SB	0.386	0.387	0.388	0.387	**0.384**	0.386	0.877	0.879	0.882	0.878	**0.872**	4.680	0.876
LA	38.228	38.234	38.246	38.253	**38.162**	38.186	129.797	129.822	129.865	129.882	**129.568**	139.121	129.650
LL	56.855	56.831	56.803	56.888	56.694	**56.694**	193.098	193.008	192.900	193.222	**192.539**	196.365	192.561
LP	144.231	144.468	144.723	145.264	**143.749**	143.807	490.338	491.202	492.051	493.945	**488.790**	567.637	488.991
GC	13.229	13.233	13.239	13.228	**13.217**	13.231	33.811	33.822	33.838	33.811	**33.780**	34.854	33.817
LN	**638.153**	638.264	638.527	638.395	638.796	638.275	**2164.044**	2164.318	2164.992	2164.800	2166.208	2196.155	2164.741
SI	0.515	0.516	0.516	0.515	0.515	**0.515**	1.202	1.202	1.202	1.202	1.201	1.253	**1.200**
LX	61.073	**61.072**	61.198	61.146	61.132	61.089	206.371	**206.357**	206.773	206.623	206.571	211.740	206.451
CL	1.681	1.681	1.684	1.682	1.679	**1.678**	5.948	5.949	5.959	5.951	5.943	6.183	**5.939**
CO	1.646	1.646	1.648	1.650	1.642	**1.642**	5.824	5.824	5.833	5.838	**5.811**	6.083	5.811
HO	4.407	4.410	4.409	4.402	4.401	**4.398**	15.616	15.626	15.624	15.599	15.597	16.627	**15.588**
PG	**3.638**	3.641	3.650	3.651	3.640	3.641	**13.142**	13.150	13.184	13.189	13.149	21.254	13.154
NG	0.160	0.160	0.161	0.160	0.160	**0.160**	0.573	0.572	0.573	0.573	0.571	1.281	**0.570**
XB	**3.725**	3.729	3.732	3.725	3.730	3.729	**12.776**	12.791	12.799	12.781	12.797	18.342	12.792

Test Period													
Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	1.462	1.464	1.461	1.464	**1.442**	1.447	4.251	4.256	4.250	4.260	**4.199**	4.753	4.211
LC	1.034	1.035	1.034	1.033	1.032	**1.031**	2.674	2.678	2.672	2.671	**2.666**	4.609	2.667
LH	0.839	0.839	0.839	0.839	**0.836**	0.839	2.390	2.391	2.390	2.389	**2.382**	6.536	2.390
C	6.852	6.853	6.857	6.884	6.842	**6.836**	16.323	16.323	16.335	16.393	16.295	22.825	**16.287**
S	13.395	13.433	13.393	13.444	13.393	**13.382**	37.189	37.307	37.176	37.321	37.173	51.402	**37.145**
W	8.931	8.931	**8.926**	8.954	8.939	8.929	21.140	21.139	**21.127**	21.195	21.158	22.098	21.136
CC	34.495	34.494	34.592	34.495	34.493	**34.467**	82.197	82.196	82.437	82.208	82.195	82.303	**82.135**
KC	3.047	3.053	3.059	3.046	3.042	**3.040**	7.171	7.186	7.200	7.170	7.159	7.165	**7.156**
KW	9.064	9.053	9.063	9.089	9.049	**9.046**	21.395	21.365	21.390	21.455	21.355	22.620	**21.347**
CT	0.845	0.846	0.852	0.845	**0.841**	0.843	2.058	2.061	2.075	2.060	**2.051**	2.326	2.055
SB	0.230	0.231	0.231	0.230	0.230	**0.230**	0.512	0.513	0.514	0.512	0.512	1.943	**0.511**
LA	20.185	20.178	20.192	20.197	**20.074**	20.094	68.408	68.384	68.431	68.447	**68.024**	69.021	68.092
LL	27.782	27.775	27.773	27.770	27.738	**27.709**	94.226	94.206	94.201	94.186	94.081	**93.950**	93.980
LP	77.589	77.556	77.649	77.902	**77.032**	77.181	262.524	262.411	262.723	263.585	**260.607**	262.281	261.145
GC	14.012	14.010	14.007	14.012	14.014	**13.987**	35.696	35.688	35.683	35.702	35.700	38.432	**35.630**
LN	235.981	**235.640**	235.695	235.903	235.874	235.748	795.593	**794.486**	794.681	795.327	795.249	799.975	794.826
SI	0.399	0.399	0.400	0.399	0.399	**0.398**	0.939	0.939	0.941	0.939	0.939	0.986	**0.938**
LX	28.891	28.884	28.920	28.967	**28.876**	28.879	97.940	97.915	98.044	98.202	**97.891**	98.190	97.903
CL	1.121	1.121	1.122	1.123	**1.112**	1.113	3.976	3.977	3.978	3.982	**3.946**	3.990	3.948
CO	1.120	1.121	1.121	1.126	1.113	**1.113**	3.970	3.971	3.974	3.989	3.945	3.989	**3.943**
HO	2.988	2.985	2.984	2.995	2.970	**2.967**	10.582	10.573	10.569	10.610	10.522	10.832	**10.509**
PG	2.876	2.878	2.881	2.892	**2.859**	2.864	10.209	10.216	10.225	10.265	**10.148**	22.135	10.166
NG	0.062	0.062	0.062	0.062	**0.061**	0.061	0.222	0.222	0.222	0.222	**0.221**	0.373	0.221
XB	2.924	2.923	2.923	2.937	2.909	**2.907**	9.958	9.951	9.950	10.004	9.904	19.216	**9.899**

Note: red bold values indicate that they are smallest in different forecasting methods.

Table 6: Relative forecasting performance evaluated using functional and multivariate error measures $RErr_f$ and $RErr_m$ for commodity futures forward curves at a daily resolution.

Validation Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	**FAR(2)**	**FAR(4)**	**FSAR**	**PF**	**Naïve**	**FAR(1)**	**FAR(2)**	**FAR(4)**	**FSAR**	**PF**	**DL**	**Naïve**
FC	0.796%	0.796%	0.796%	0.798%	***0.791%***	0.795%	2.130%	2.129%	2.129%	2.133%	***2.116%***	2.394%	2.126%
LC	0.780%	0.780%	0.781%	0.781%	***0.780%***	0.780%	1.950%	1.949%	1.952%	1.953%	***1.948%***	3.514%	1.948%
LH	1.164%	1.163%	1.163%	1.163%	***1.160%***	1.163%	3.191%	3.190%	3.190%	3.190%	***3.188%***	7.503%	3.191%
C	2.030%	2.029%	2.030%	2.033%	2.026%	***2.025%***	4.586%	4.584%	4.588%	4.594%	4.576%	5.093%	***4.575%***
S	1.659%	***1.658%***	1.661%	1.662%	1.659%	1.658%	4.400%	***4.397%***	4.405%	4.408%	4.399%	4.899%	4.398%
W	2.203%	2.202%	2.203%	2.209%	2.201%	***2.200%***	5.000%	4.997%	5.000%	5.013%	4.994%	5.787%	***4.992%***
CC	1.839%	1.840%	1.841%	1.841%	1.839%	***1.838%***	4.138%	4.139%	4.141%	4.142%	4.137%	4.145%	***4.135%***
KC	***1.756%***	1.761%	1.762%	1.757%	1.757%	1.757%	***3.953%***	3.963%	3.966%	3.954%	3.954%	3.986%	3.953%
KW	2.050%	2.049%	2.050%	2.050%	2.046%	***2.044%***	4.622%	4.619%	4.621%	4.621%	4.614%	5.375%	***4.608%***
CT	1.809%	1.812%	1.813%	1.811%	1.808%	***1.807%***	4.082%	4.088%	4.089%	4.086%	4.079%	4.639%	***4.078%***
SB	2.228%	2.235%	2.241%	2.234%	***2.220%***	2.226%	4.561%	4.575%	4.587%	4.574%	***4.543%***	20.847%	4.556%
LA	1.666%	1.667%	1.667%	1.666%	1.664%	***1.664%***	5.549%	5.551%	5.552%	5.548%	5.540%	5.871%	***5.540%***
LL	2.786%	2.785%	2.784%	2.786%	2.778%	***2.778%***	9.250%	9.245%	9.242%	9.251%	9.223%	9.360%	***9.222%***
LP	2.294%	2.297%	2.302%	2.308%	***2.284%***	2.286%	7.612%	7.623%	7.640%	7.659%	***7.581%***	8.458%	7.587%
GC	1.398%	1.398%	1.398%	***1.397%***	1.397%	1.397%	3.422%	3.420%	3.422%	***3.418%***	3.418%	3.552%	3.420%
LN	2.846%	2.845%	***2.844%***	2.848%	2.847%	2.845%	9.415%	9.412%	***9.406%***	9.421%	9.418%	9.490%	9.410%
SI	2.590%	2.589%	2.589%	2.592%	2.587%	***2.585%***	5.788%	5.786%	5.788%	5.792%	5.781%	6.198%	***5.778%***
LX	2.579%	2.578%	2.584%	2.580%	2.580%	***2.578%***	8.538%	8.536%	8.556%	8.543%	8.542%	8.660%	***8.535%***
CL	2.141%	2.143%	2.149%	2.141%	2.138%	***2.136%***	7.500%	7.507%	7.533%	7.501%	7.492%	7.741%	***7.485%***
CO	2.037%	2.038%	2.044%	2.040%	2.030%	***2.029%***	7.128%	7.132%	7.155%	7.138%	7.104%	7.377%	***7.101%***
HO	1.955%	1.956%	1.958%	1.952%	1.948%	***1.946%***	6.826%	6.831%	6.836%	6.816%	6.802%	7.267%	***6.796%***
PG	1.399%	1.400%	1.404%	1.404%	***1.397%***	1.398%	4.860%	4.862%	4.875%	4.877%	***4.854%***	7.807%	4.857%
NG	2.187%	2.186%	2.190%	2.188%	2.182%	***2.178%***	7.975%	7.972%	7.982%	7.972%	7.951%	16.751%	***7.936%***
XB	1.500%	1.501%	1.501%	***1.498%***	1.500%	1.499%	5.021%	5.024%	5.027%	***5.016%***	5.022%	7.111%	5.019%

Test Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	**FAR(2)**	**FAR(4)**	**FSAR**	**PF**	**Naïve**	**FAR(1)**	**FAR(2)**	**FAR(4)**	**FSAR**	**PF**	**DL**	**Naïve**
FC	1.001%	1.002%	1.001%	1.003%	***0.989%***	0.990%	2.656%	2.659%	2.654%	2.660%	***2.625%***	2.951%	2.627%
LC	0.850%	0.852%	0.850%	0.850%	0.849%	***0.848%***	2.104%	2.107%	2.102%	2.101%	2.097%	3.597%	***2.097%***
LH	1.108%	1.109%	1.108%	1.108%	***1.105%***	1.108%	3.024%	3.025%	3.025%	3.022%	***3.017%***	8.063%	3.024%
C	1.421%	1.422%	1.422%	1.425%	1.420%	***1.418%***	3.201%	3.201%	3.203%	3.209%	3.198%	4.077%	***3.194%***
S	1.180%	1.182%	1.180%	1.183%	1.179%	***1.178%***	3.138%	3.144%	3.138%	3.147%	3.136%	4.149%	***3.133%***
W	1.512%	1.512%	1.511%	1.516%	1.513%	***1.511%***	3.435%	3.435%	3.434%	3.444%	3.437%	3.531%	***3.433%***
CC	1.457%	1.457%	1.461%	1.457%	1.456%	***1.455%***	3.291%	3.291%	3.301%	3.292%	3.290%	3.291%	***3.287%***
KC	1.934%	1.938%	1.940%	1.933%	1.931%	***1.931%***	4.353%	4.363%	4.368%	4.353%	4.348%	4.351%	***4.346%***
KW	1.469%	1.468%	1.469%	1.473%	1.467%	***1.466%***	3.322%	3.320%	3.322%	3.331%	3.318%	3.470%	***3.316%***
CT	1.181%	1.182%	1.189%	1.181%	***1.176%***	1.178%	2.712%	2.717%	2.732%	2.715%	***2.703%***	3.004%	2.708%
SB	1.403%	1.403%	1.406%	1.403%	1.404%	***1.400%***	2.907%	2.908%	2.914%	2.907%	2.907%	11.470%	***2.900%***
LA	1.085%	1.084%	1.085%	1.085%	***1.079%***	1.080%	3.609%	3.608%	3.610%	3.612%	***3.589%***	3.644%	3.593%
LL	1.381%	1.381%	1.381%	1.381%	1.379%	***1.378%***	4.585%	4.584%	4.583%	4.583%	4.578%	4.574%	***4.572%***
LP	1.226%	1.225%	1.226%	1.229%	***1.217%***	1.219%	4.065%	4.064%	4.068%	4.077%	***4.035%***	4.058%	4.041%
GC	1.058%	1.058%	1.057%	1.058%	1.058%	***1.055%***	2.586%	2.586%	2.586%	2.587%	2.587%	2.771%	***2.581%***
LN	1.748%	***1.746%***	1.747%	1.748%	1.747%	1.746%	5.787%	***5.782%***	5.785%	5.787%	5.786%	5.856%	5.782%
SI	1.794%	1.793%	1.796%	1.792%	1.793%	***1.790%***	4.006%	4.004%	4.010%	4.001%	4.004%	4.243%	***3.998%***
LX	1.367%	1.367%	1.368%	1.370%	1.366%	***1.366%***	4.541%	4.540%	4.545%	4.550%	4.537%	4.542%	***4.537%***
CL	1.821%	1.821%	1.821%	1.823%	***1.804%***	1.805%	6.367%	6.370%	6.368%	6.375%	***6.309%***	6.362%	6.313%
CO	1.719%	1.719%	1.721%	1.726%	***1.706%***	1.707%	6.009%	6.010%	6.017%	6.036%	***5.967%***	6.024%	5.968%
HO	1.563%	1.562%	1.561%	1.565%	1.553%	***1.551%***	5.445%	5.444%	5.441%	5.456%	5.411%	5.546%	***5.407%***
PG	1.842%	1.842%	1.844%	1.849%	***1.831%***	1.835%	6.447%	6.448%	6.454%	6.471%	***6.410%***	13.812%	6.422%
NG	1.838%	1.837%	1.836%	1.836%	***1.828%***	1.828%	6.599%	6.600%	6.594%	6.592%	***6.563%***	11.514%	6.564%
XB	1.806%	1.804%	1.803%	1.813%	1.795%	***1.795%***	6.078%	6.070%	6.065%	6.102%	6.041%	11.566%	***6.039%***

Note: red bold values indicate that they are smallest in different forecasting methods.

sequence of curves forming a weak white noise and exhibiting conditional heteroscedasticity. We applied several functional time series forecasting methods to the log-differenced forward curves, as well as the multivariate method of Diebold & Li (2006), and found that in general the functional methods were superior to their multivariate counterparts, and in some cases even beat the naïve forecast in test period comparisons. Among the functional forecasting methods, the FAR(1) model estimated using predictive factors Kargin & Onatski (2008) performed slightly better than the alternatives and benchmarks considered.

Appendix A Additional technical information

A.1 Implementation of functional KPSS and portmanteau tests

As discussed in Section 3, the functional KPSS statistic defined as

$$\text{fKPSS}_N = \iint Z_N^2(x,t)dtdx$$

satisfies under mild weak dependence and moment conditions on the log-differenced series $X_i(t)$ that

$$\text{fKPSS}_N \xrightarrow{D} \Xi_{KPSS}, \ N \to \infty,$$

where Ξ_{KPSS} is a random variable that can be expressed in terms of the eigenvalues of the long-run covariance operator

$$C_{LR}(x)(t) = \int c_{LR}(t,s)x(s)ds,$$

with

$$c_{LR}(t,s) = \sum_{\ell=-\infty}^{\infty} \text{cov}(X_0(t), X_\ell(s)),$$

see Horváth et al. (2014) for details. In order to estimate these eigenvalues and subsequently the limiting distribution, we used a kernel lag-window estimator of c_{LR} defined by

$$\hat{c}_{LR}(t,s) = \sum_{\ell=-\infty}^{\infty} K(\ell/h)\hat{c}_\ell(t,s),$$

with a flat-top kernel

$$K(x) = \begin{cases} 1, & \text{if } 0 \le t < 1 \\ 1.1 - |t|, & \text{if } 0.1 \le t < 1.1 \\ 0, & \text{if } |t| \ge 1.1 \end{cases}$$

and bandwidth $h = N^{1/2}$.

Regarding the portmanteau tests, under the conditions of Theorem 2 in Kokoszka et al. (2017), we have under $\mathcal{H}_{0,iid}$ and $\mathcal{H}_{0,ch}$, respectively, that

$$V_{N,K} \xrightarrow{D} \Theta_{K,iid}, \ \ N \to \infty,$$

and

$$V_{N,K} \xrightarrow{D} \Theta_{K,ch}, \quad N \to \infty.$$

Both $\Theta_{K,iid}$ and $\Theta_{K,ch}$ can be approximated using a Welch style approximation of the form

$$\Theta_{K,iid} \overset{D}{\approx} \beta_{iid}\chi^2_{\nu_{iid}},$$

and

$$\Theta_{K,ch} \overset{D}{\approx} \beta_{ch}\chi^2_{\nu_{ch}},$$

where the constants β and ν are estimated in order that the scaled chi-square approximations share approximately the same first two moments with the limiting distribution. Estimating β_{ch} and ν_{ch} is discussed in detail in Section 2 of Kokoszka et al. (2017), and amounts to taking

$$\hat{\beta}_{ch} = \frac{\hat{\sigma}^2_{K,ch}}{2\hat{\mu}_{K,ch}} \quad \text{and} \quad \hat{\nu}_{ch} = \frac{2\hat{\mu}^2_{K,ch}}{\hat{\sigma}^2_{K,ch}}, \tag{9}$$

where $\hat{\mu}_{K,ch}$ and $\hat{\sigma}^2_{K,ch}$ are estimators of the mean and variance of $\Theta_{K,ch}$. It follows by examining equation (15) in Kokoszka et al. (2017) that under $\mathcal{H}_{0,iid}$ as opposed to $\mathcal{H}_{0,ch}$,

$$E\Theta_{K,iid} = K\left(\int \text{cov}(X_0(t), X_0(t))dt\right)^2, \ \text{Var}(\Theta_{K,iid}) = 2K\left(\iint \text{cov}(X_0(t), X_0(s))dtds\right)^2,$$

which can be similarly estimated, yielding estimates of β_{iid} and ν_{iid}.

A.2 Estimation of FSAR models

We estimate these models using functional principal component analysis and the least squares principle: Let $\hat{v}_1, ..., \hat{v}_p$ be the first p functional principal components. Let $\ell_1, ..., \ell_k$ denote the pre-specified lags. We assume that the kernels $\psi_{\ell,i}$ can be approximated by

$$\psi_{\ell,i}(t,s) \approx \sum_{j,r=1}^{p} \mathbf{\Phi}_{\ell,i}[j,r]\hat{v}_j(t)\hat{v}_r(s), \quad \mathbf{\Phi}_i \in \mathbb{R}^{p\times p},$$

and further that

$$X_n(t) \approx \sum_{j=1}^{p} \langle X_n, \hat{v}_j\rangle \hat{v}_j(t), \quad \mathbf{X}_n = (\langle X_n, \hat{v}_1, \rangle, \ldots, \langle X_n, \hat{v}_d\rangle)^\top \in \mathbb{R}^p.$$

Let

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}_{\ell,1} \\ \vdots \\ \mathbf{\Phi}_{\ell,k} \end{pmatrix} \in \mathbb{R}^{kp\times p},$$

Rewriting the FSAR equations using these approximations, we have that with

$$\mathbf{X}_{L,p} = \Big(\ \mathbf{X}_{i-\ell_1}^{\top}, ..., \mathbf{X}_{i-\ell_k}^{\top}, \ i = \ell_k + 1, ..., N \ \Big)^{\top} \in \mathbb{R}^{N-\ell_k \times kp},$$

$$\mathbf{X}_{R,p} = \begin{pmatrix} \mathbf{X}_{\ell_k+1}^{\top} \\ \vdots \\ \mathbf{X}_{N}^{\top} \end{pmatrix} \in \mathbb{R}^{N-\ell_k \times p},$$

the least squares estimator for $\mathbf{\Phi}$ is

$$\widehat{\mathbf{\Phi}} = (\mathbf{X}_{L,p}^{\top}\mathbf{X}_{L,p})^{-1}\mathbf{X}_{L,p}\mathbf{X}_{R,p} =: \begin{pmatrix} \hat{\mathbf{\Phi}}_{\ell,1} \\ \vdots \\ \hat{\mathbf{\Phi}}_{\ell,k} \end{pmatrix} \in \mathbb{R}^{kp \times p}.$$

This yields estimates of the kernel functions $\psi_{\ell,i}(t,s)$ given by

$$\hat{\psi}_{\ell,i}(t,s) \approx \sum_{j,r=1}^{p} \hat{\mathbf{\Phi}}_{\ell,i}[j,r]\hat{v}_j(t)\hat{v}_r(s), \quad \mathbf{\Phi}_i \in \mathbb{R}^{p \times p}.$$

Appendix B Results for forecasting at different temporal resolutions

Table 7: Forecasting performance evaluated using functional and multivariate error measures for commodity futures forward curves at a weekly resolution.

Validation Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	0.799	0.798	0.798	0.799	***0.793***	0.798	2.334	2.332	2.332	2.333	***2.318***	2.612	2.331
LC	***0.737***	0.738	0.739	0.739	0.738	0.738	***1.898***	1.899	1.902	1.901	1.898	3.339	1.899
LH	0.848	0.848	0.848	0.848	***0.847***	0.848	2.383	2.383	2.383	2.382	***2.379***	5.362	2.383
C	9.262	9.259	9.265	9.270	***9.246***	9.249	22.007	22.006	22.018	22.027	***21.966***	25.480	21.975
S	17.300	***17.290***	17.315	17.332	17.296	17.293	47.715	***47.687***	47.761	47.807	47.702	53.755	47.692
W	15.228	15.206	15.215	15.227	15.199	***15.198***	36.040	35.991	36.016	36.043	***35.966***	42.133	35.971
CC	45.136	45.150	45.183	45.237	45.140	***45.124***	106.656	106.697	106.780	106.902	106.665	106.966	***106.625***
KC	***2.770***	2.779	2.778	2.773	2.774	2.774	***6.521***	6.543	6.541	6.531	6.532	6.605	6.532
KW	14.744	14.737	14.737	14.720	***14.657***	14.678	34.833	34.821	34.826	34.766	***34.636***	40.150	34.690
CT	1.620	1.620	***1.616***	1.620	1.617	1.618	3.869	3.868	***3.856***	3.870	3.862	4.616	3.865
SB	0.388	0.389	0.390	0.389	***0.386***	0.388	0.879	0.881	0.884	0.880	***0.874***	4.680	0.878
LA	38.395	38.401	38.413	38.422	***38.334***	38.356	129.781	129.810	129.853	129.873	***129.573***	139.121	129.649
LL	57.115	57.097	57.066	57.151	***56.952***	56.953	193.067	193.008	192.896	193.194	***192.510***	196.365	192.528
LP	144.956	145.194	145.447	145.974	***144.442***	144.504	490.402	491.227	492.069	493.917	***488.726***	567.637	488.933
GC	13.288	13.292	13.298	13.287	***13.276***	13.290	33.812	33.824	33.839	33.811	***33.781***	34.854	33.818
LN	***641.845***	641.863	642.030	642.143	642.524	642.004	2165.755	***2165.722***	2166.075	2166.677	2168.029	2196.155	2166.579
SI	0.517	0.518	0.518	0.517	0.517	***0.517***	1.202	1.202	1.202	1.202	1.201	1.253	***1.200***
LX	61.349	***61.348***	61.478	61.432	61.411	61.367	206.402	***206.390***	206.821	206.686	206.608	211.740	206.489
CL	1.687	1.687	1.690	1.688	1.685	***1.684***	5.952	5.953	5.963	5.955	5.947	6.183	***5.943***
CO	1.651	1.651	1.654	1.655	1.648	***1.648***	5.824	5.824	5.835	5.840	***5.814***	6.083	5.814
HO	4.421	4.425	4.424	4.418	4.415	***4.413***	15.621	15.636	15.632	15.610	15.605	16.627	***15.595***
PG	***3.674***	3.675	3.685	3.685	3.674	3.676	***13.221***	13.226	13.259	13.265	13.225	21.254	13.231
NG	0.163	0.163	0.163	0.163	0.162	***0.162***	0.580	0.580	0.580	0.580	0.578	1.281	***0.577***
XB	***3.765***	3.768	3.770	3.766	3.769	3.768	***12.853***	12.862	12.870	12.858	12.869	18.342	12.864

Test Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	1.472	1.474	1.472	1.475	***1.452***	1.458	4.272	4.275	4.271	4.281	***4.218***	4.753	4.232
LC	1.047	1.047	1.047	1.047	1.045	***1.045***	2.694	2.691	2.691	2.690	***2.686***	4.609	2.686
LH	0.880	0.881	0.881	***0.880***	0.880	0.880	2.490	2.491	2.491	2.489	***2.488***	6.536	2.491
C	6.917	6.917	6.922	6.938	6.909	***6.902***	16.394	16.392	16.403	16.442	16.375	22.825	***16.363***
S	13.501	13.528	13.500	13.540	13.500	***13.489***	37.575	37.668	37.566	37.679	37.567	51.402	***37.538***
W	8.973	8.973	***8.968***	8.993	8.981	8.971	21.176	21.175	***21.163***	21.225	21.194	22.098	21.171
CC	34.637	34.639	34.741	34.640	34.637	***34.612***	82.231	82.233	82.484	82.246	82.233	82.303	***82.173***
KC	3.054	3.065	3.069	3.054	3.050	***3.049***	7.164	7.190	7.199	7.165	7.155	7.165	***7.152***
KW	9.102	9.095	9.106	9.127	9.091	***9.088***	21.422	21.403	21.430	21.487	21.395	22.620	***21.386***
CT	0.849	0.849	0.857	0.849	***0.845***	0.847	2.060	2.062	2.079	2.060	***2.051***	2.326	2.055
SB	0.231	0.232	0.232	0.231	0.231	***0.231***	0.513	0.513	0.514	0.513	0.512	1.943	***0.511***
LA	20.331	20.323	20.336	20.342	***20.217***	20.242	68.541	68.511	68.557	68.577	***68.152***	69.021	68.232
LL	27.927	27.933	27.936	27.915	27.887	***27.854***	94.232	94.250	94.260	94.194	94.103	***93.950***	93.989
LP	78.075	78.058	78.076	78.329	***77.456***	77.609	262.753	262.693	262.755	263.615	***260.648***	262.281	261.197
GC	14.075	14.073	14.071	14.082	14.077	***14.050***	35.696	35.689	35.683	35.721	35.701	38.432	***35.630***
LN	237.134	***236.785***	236.839	237.048	237.029	236.901	795.470	***794.348***	794.536	795.176	795.131	799.975	794.709
SI	0.401	0.401	0.401	0.400	0.401	***0.400***	0.939	0.939	0.941	0.939	0.939	0.986	***0.938***
LX	29.041	29.034	29.071	29.117	***29.026***	29.029	97.937	97.913	98.041	98.197	***97.888***	98.190	97.900
CL	1.126	1.126	1.127	1.128	***1.117***	1.118	3.982	3.983	3.984	3.989	***3.952***	3.990	3.954
CO	1.125	1.126	1.126	1.131	1.118	***1.118***	3.974	3.976	3.978	3.994	3.949	3.989	***3.948***
HO	2.997	2.995	2.994	3.002	2.980	***2.977***	10.589	10.583	10.578	10.609	10.531	10.832	***10.518***
PG	2.919	2.922	2.925	2.936	***2.903***	2.908	10.304	10.317	10.325	10.365	***10.247***	22.135	10.265
NG	0.062	0.062	0.062	0.062	***0.062***	0.062	0.223	0.223	0.223	0.223	***0.222***	0.373	0.222
XB	2.968	2.965	2.965	2.976	2.951	***2.949***	10.050	10.040	10.040	10.077	9.992	19.216	***9.987***

Note: red bold values indicate that they are smallest in different forecasting methods.

Table 8: Relative forecasting performance evaluated using functional and multivariate error measures for commodity futures forward curves at a weekly resolution.

Validation Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	0.799%	0.798%	0.799%	0.800%	**0.794%**	0.798%	2.137%	2.135%	2.136%	2.138%	**2.125%**	2.394%	2.134%
LC	**0.786%**	0.786%	0.788%	0.787%	0.786%	0.786%	**1.961%**	1.962%	1.965%	1.964%	1.961%	3.514%	1.961%
LH	1.201%	1.201%	1.201%	1.201%	**1.198%**	1.201%	3.302%	3.301%	3.301%	3.302%	**3.297%**	7.503%	3.301%
C	2.031%	2.030%	2.031%	2.033%	2.027%	**2.027%**	4.590%	4.589%	4.592%	4.596%	4.581%	5.093%	**4.580%**
S	1.660%	**1.659%**	1.662%	1.663%	1.660%	1.660%	4.407%	**4.405%**	4.413%	4.413%	4.407%	4.899%	4.406%
W	2.205%	2.204%	2.205%	2.210%	2.203%	**2.202%**	5.007%	5.004%	5.008%	5.019%	5.001%	5.787%	**4.999%**
CC	1.839%	1.840%	1.841%	1.841%	1.839%	**1.838%**	4.139%	4.142%	4.144%	4.144%	4.140%	4.145%	**4.137%**
KC	**1.759%**	1.765%	1.765%	1.759%	1.760%	1.759%	**3.961%**	3.974%	3.975%	3.962%	3.962%	3.986%	3.961%
KW	2.051%	2.050%	2.051%	2.051%	2.048%	**2.045%**	4.625%	4.622%	4.625%	4.624%	4.617%	5.375%	**4.611%**
CT	1.812%	1.814%	1.815%	1.813%	1.810%	**1.809%**	4.101%	4.106%	4.106%	4.104%	4.097%	4.639%	**4.096%**
SB	2.230%	2.236%	2.242%	2.236%	**2.222%**	2.227%	4.570%	4.583%	4.594%	4.584%	**4.554%**	20.847%	4.564%
LA	1.666%	1.667%	1.667%	1.666%	1.664%	**1.664%**	5.551%	5.552%	5.554%	5.550%	5.543%	5.871%	**5.542%**
LL	2.786%	2.784%	2.784%	2.786%	2.778%	**2.777%**	9.250%	9.246%	9.245%	9.253%	9.223%	9.360%	**9.223%**
LP	2.294%	2.298%	2.302%	2.307%	**2.284%**	2.285%	7.616%	7.628%	7.641%	7.658%	**7.580%**	8.458%	7.586%
GC	1.398%	1.398%	1.399%	**1.396%**	1.397%	1.398%	3.422%	3.421%	3.423%	**3.417%**	3.419%	3.552%	3.420%
LN	2.848%	2.847%	**2.845%**	2.850%	2.849%	2.846%	9.421%	9.417%	**9.411%**	9.428%	9.424%	9.490%	9.416%
SI	2.589%	2.589%	2.589%	2.592%	2.587%	**2.585%**	5.786%	5.787%	5.788%	5.794%	5.782%	6.198%	**5.779%**
LX	2.578%	2.578%	2.584%	2.580%	2.580%	**2.578%**	8.539%	8.537%	8.557%	8.545%	8.544%	8.660%	**8.537%**
CL	2.140%	2.142%	2.149%	2.141%	2.138%	**2.136%**	7.507%	7.514%	7.540%	7.508%	7.499%	7.741%	**7.493%**
CO	2.037%	2.038%	2.045%	2.041%	2.031%	**2.030%**	7.135%	7.139%	7.165%	7.148%	7.115%	7.377%	**7.112%**
HO	1.955%	1.957%	1.959%	1.952%	1.948%	**1.947%**	6.829%	6.838%	6.843%	6.820%	6.808%	7.267%	**6.802%**
PG	1.410%	1.410%	1.414%	1.414%	**1.408%**	1.408%	4.888%	4.889%	4.903%	4.904%	**4.881%**	7.807%	4.884%
NG	2.206%	2.205%	2.207%	2.205%	2.200%	**2.195%**	8.060%	8.064%	8.065%	8.055%	8.035%	16.751%	**8.019%**
XB	1.509%	1.510%	1.510%	**1.508%**	1.509%	1.508%	5.048%	5.048%	5.051%	**5.042%**	5.047%	7.111%	5.044%

Test Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	1.007%	1.007%	1.006%	1.008%	**0.994%**	0.996%	2.669%	2.671%	2.667%	2.674%	**2.636%**	2.951%	2.640%
LC	0.859%	0.859%	0.859%	0.859%	0.858%	**0.857%**	2.118%	2.116%	2.116%	2.116%	2.113%	3.597%	**2.112%**
LH	1.157%	1.157%	1.157%	**1.156%**	1.156%	1.156%	3.149%	3.150%	3.149%	**3.145%**	3.147%	8.063%	3.148%
C	1.428%	1.428%	1.428%	1.430%	1.427%	**1.425%**	3.212%	3.212%	3.214%	3.218%	3.210%	4.077%	**3.206%**
S	1.184%	1.186%	1.185%	1.187%	1.184%	**1.183%**	3.164%	3.169%	3.165%	3.171%	3.162%	4.149%	**3.159%**
W	1.515%	1.515%	1.514%	1.518%	1.516%	**1.514%**	3.443%	3.443%	3.441%	3.450%	3.445%	3.531%	**3.441%**
CC	1.457%	1.456%	1.461%	1.457%	1.456%	**1.455%**	3.292%	3.292%	3.303%	3.293%	3.292%	3.291%	**3.288%**
KC	1.931%	1.937%	1.939%	1.931%	1.929%	**1.928%**	4.350%	4.365%	4.369%	4.351%	4.346%	4.351%	**4.344%**
KW	1.471%	1.471%	1.472%	1.475%	1.470%	**1.469%**	3.329%	3.328%	3.331%	3.338%	3.326%	3.470%	**3.325%**
CT	1.181%	1.182%	1.191%	1.181%	**1.177%**	1.178%	2.715%	2.717%	2.736%	2.715%	**2.704%**	3.004%	2.708%
SB	1.404%	1.405%	1.407%	1.404%	1.404%	**1.401%**	2.909%	2.910%	2.915%	2.910%	2.909%	11.470%	**2.903%**
LA	1.086%	1.086%	1.087%	1.087%	**1.080%**	1.082%	3.615%	3.613%	3.616%	3.618%	**3.595%**	3.644%	3.600%
LL	1.381%	1.381%	1.381%	1.380%	1.379%	**1.377%**	4.585%	4.585%	4.585%	4.583%	4.579%	4.574%	**4.573%**
LP	1.226%	1.226%	1.226%	1.229%	**1.217%**	1.219%	4.067%	4.067%	4.068%	4.078%	**4.035%**	4.058%	4.042%
GC	1.058%	1.058%	1.057%	1.059%	1.058%	**1.055%**	2.586%	2.586%	2.586%	2.589%	2.587%	2.771%	**2.581%**
LN	1.748%	**1.746%**	1.747%	1.747%	1.747%	1.746%	5.786%	**5.780%**	5.784%	5.786%	5.784%	5.856%	5.781%
SI	1.793%	1.793%	1.795%	1.791%	1.793%	**1.790%**	4.004%	4.004%	4.009%	4.000%	4.004%	4.243%	**3.998%**
LX	1.367%	1.367%	1.368%	1.370%	1.366%	**1.366%**	4.540%	4.540%	4.544%	4.550%	4.537%	4.542%	**4.537%**
CL	1.823%	1.823%	1.823%	1.825%	**1.806%**	1.807%	6.380%	6.383%	6.381%	6.388%	**6.322%**	6.362%	6.326%
CO	1.720%	1.721%	1.722%	1.727%	**1.708%**	1.708%	6.018%	6.020%	6.026%	6.044%	**5.975%**	6.024%	5.977%
HO	1.563%	1.563%	1.562%	1.566%	1.553%	**1.552%**	5.451%	5.449%	5.447%	5.461%	5.417%	5.546%	**5.412%**
PG	1.862%	1.863%	1.865%	1.870%	**1.852%**	1.855%	6.505%	6.510%	6.516%	6.532%	**6.471%**	13.812%	6.483%
NG	1.850%	1.850%	1.849%	1.850%	1.841%	**1.841%**	6.647%	6.651%	6.644%	6.648%	6.611%	11.514%	**6.611%**
XB	1.826%	1.823%	1.822%	1.829%	1.814%	**1.813%**	6.131%	6.119%	6.115%	6.137%	6.089%	11.566%	**6.087%**

Note: red bold values indicate that they are smallest in different forecasting methods.

Table 9: Forecasting performance evaluated using functional and multivariate error measures for commodity futures forward curves at a monthly resolution.

Validation Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	0.811	0.811	0.811	0.813	**0.806**	0.810	2.362	2.361	2.360	2.365	**2.347**	2.612	2.359
LC	0.788	0.788	0.789	0.789	**0.788**	0.788	1.999	1.999	2.001	2.001	**1.999**	3.339	1.999
LH	1.003	1.002	1.003	**0.998**	1.002	1.002	2.883	2.882	2.884	**2.868**	2.882	5.362	2.883
C	9.279	9.273	9.278	9.300	**9.266**	9.269	22.156	22.145	22.158	22.212	**22.124**	25.480	22.129
S	17.253	**17.216**	17.261	17.284	17.244	17.240	47.804	**47.731**	47.839	47.895	47.783	53.755	47.770
W	15.291	15.266	15.277	15.261	15.259	**15.257**	36.401	36.359	36.392	36.344	**36.318**	42.133	36.320
CC	45.103	45.116	45.136	45.191	45.108	**45.090**	107.016	107.056	107.113	107.236	107.028	**106.966**	106.987
KC	**2.785**	2.792	2.791	2.790	2.789	2.789	**6.543**	6.560	6.560	6.557	6.554	6.605	6.553
KW	14.752	14.748	14.750	14.729	**14.674**	14.692	35.068	35.067	35.072	35.000	**34.894**	40.150	34.938
CT	1.671	1.673	**1.664**	1.671	1.670	1.669	4.021	4.024	**4.001**	4.021	4.017	4.616	4.016
SB	0.401	0.402	0.403	0.402	**0.400**	0.400	0.900	0.902	0.905	0.902	**0.897**	4.680	0.898
LA	38.359	38.368	38.380	38.376	**38.292**	38.313	130.015	130.056	130.102	130.074	**129.796**	139.121	129.867
LL	56.938	56.896	56.873	56.982	56.780	**56.776**	193.100	192.943	192.842	193.254	**192.554**	196.365	192.563
LP	144.562	144.844	145.132	145.682	**143.990**	144.115	490.981	491.997	492.974	494.892	**489.251**	567.637	489.671
GC	13.483	13.487	13.494	13.482	**13.470**	13.484	33.833	33.845	33.863	33.831	**33.800**	34.854	33.837
LN	642.054	**641.989**	642.176	642.469	642.829	642.288	2167.377	**2167.036**	2167.389	2168.658	2169.989	2196.155	2168.522
SI	0.525	0.525	0.525	0.525	0.524	**0.524**	1.202	1.202	1.203	1.202	1.201	1.253	**1.201**
LX	**61.565**	61.568	61.677	61.652	61.634	61.587	206.857	**206.856**	207.214	207.162	207.099	211.740	206.982
CL	1.698	1.698	1.700	1.698	1.696	**1.695**	5.965	5.966	5.974	5.965	5.959	6.183	**5.956**
CO	1.647	1.647	1.648	1.650	1.644	**1.644**	5.828	5.828	5.833	5.839	5.817	6.083	**5.816**
HO	4.518	4.520	4.519	4.517	4.511	**4.508**	15.716	15.723	15.718	15.711	15.691	16.627	**15.682**
PG	3.841	3.843	3.854	3.854	**3.840**	3.843	13.824	13.826	13.865	13.868	**13.817**	21.254	13.828
NG	0.182	0.182	0.182	0.182	0.181	**0.181**	0.640	0.640	0.640	0.641	0.640	1.281	**0.638**
XB	3.999	4.000	4.005	**3.999**	4.003	4.002	13.535	13.538	13.550	**13.532**	13.548	18.342	13.542

Test Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	1.489	1.490	1.489	1.491	**1.468**	1.474	4.317	4.320	4.318	4.325	**4.262**	4.753	4.275
LC	1.108	1.108	1.109	1.109	1.107	**1.106**	2.831	2.833	2.833	2.833	2.829	4.609	**2.828**
LH	1.113	1.113	1.112	**1.105**	1.113	1.113	3.183	3.184	3.181	**3.160**	3.183	6.536	3.183
C	7.172	7.172	7.176	7.188	7.159	**7.154**	17.251	17.252	17.264	17.283	17.212	22.825	**17.200**
S	14.101	14.149	14.109	14.144	14.104	**14.091**	40.285	40.462	40.303	40.349	40.285	51.402	**40.245**
W	8.970	8.969	**8.965**	8.995	8.977	8.967	21.281	21.280	**21.271**	21.341	21.297	22.098	21.275
CC	34.477	34.478	34.561	34.478	34.477	**34.452**	82.256	82.259	82.466	82.269	82.258	82.303	**82.197**
KC	3.072	3.082	3.084	3.071	3.067	**3.066**	7.197	7.220	7.226	7.195	7.187	**7.165**	7.184
KW	9.083	9.076	9.086	9.101	9.072	**9.070**	21.492	21.475	21.498	21.534	21.464	22.620	**21.458**
CT	0.848	0.849	0.855	0.849	0.847	**0.847**	2.078	2.079	2.095	2.079	2.075	2.326	**2.074**
SB	0.235	0.236	0.236	0.235	0.235	**0.235**	0.515	0.515	0.516	0.515	0.515	1.943	**0.513**
LA	20.335	20.328	20.340	20.350	**20.223**	20.247	68.829	68.805	68.846	68.881	**68.440**	69.021	68.521
LL	27.799	27.801	27.802	27.791	27.787	**27.732**	94.218	94.228	94.232	94.196	94.180	**93.950**	93.990
LP	77.716	77.712	77.805	78.066	**77.114**	77.323	262.505	262.492	262.793	263.692	**260.398**	262.281	261.171
GC	14.264	14.262	14.260	14.265	14.267	**14.240**	35.713	35.706	35.702	35.720	35.720	38.432	**35.649**
LN	236.250	**235.929**	235.993	236.187	236.189	236.065	795.819	**794.784**	795.006	795.583	795.618	799.975	795.193
SI	0.406	0.406	0.407	0.406	0.406	**0.406**	0.940	0.940	0.941	0.939	0.940	0.986	**0.938**
LX	28.941	28.933	28.967	29.012	**28.923**	28.926	98.055	98.027	98.151	98.300	**97.997**	98.190	98.009
CL	1.130	1.130	1.131	1.132	1.123	**1.123**	3.988	3.989	3.990	3.994	3.964	3.990	**3.963**
CO	1.122	1.122	1.123	1.127	1.116	**1.115**	3.965	3.967	3.969	3.985	3.946	3.989	**3.943**
HO	3.055	3.056	3.055	3.059	3.041	**3.037**	10.673	10.675	10.671	10.687	10.627	10.832	**10.611**
PG	3.282	3.283	3.285	3.296	**3.260**	3.273	11.513	11.516	11.525	11.561	**11.439**	22.135	11.480
NG	0.066	0.066	0.066	0.066	0.066	**0.066**	0.233	0.233	0.234	0.234	0.233	0.373	**0.233**
XB	3.333	3.333	3.333	3.341	3.322	**3.320**	11.221	11.221	11.221	11.248	11.185	19.216	**11.180**

Note: red bold values indicate that they are smallest in different forecasting methods.

Table 10: Relative forecasting performance evaluated using functional and multivariate error measures for commodity futures forward curves at a monthly resolution.

Validation Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	0.811%	0.811%	0.811%	0.813%	***0.807%***	0.810%	2.164%	2.164%	2.163%	2.169%	***2.152%***	2.394%	2.161%
LC	0.826%	0.826%	0.827%	0.827%	0.826%	***0.826%***	2.070%	2.070%	2.073%	2.072%	2.070%	3.514%	***2.070%***
LH	1.426%	1.426%	1.427%	***1.420%***	1.426%	1.426%	3.992%	3.991%	3.994%	***3.975%***	3.993%	7.503%	3.993%
C	2.051%	2.049%	2.051%	2.055%	2.047%	***2.047%***	4.629%	4.626%	4.630%	4.640%	4.620%	5.093%	***4.619%***
S	1.667%	***1.664%***	1.669%	1.670%	1.667%	1.666%	4.423%	***4.415%***	4.427%	4.431%	4.422%	4.899%	4.420%
W	2.233%	2.232%	2.234%	2.234%	2.230%	***2.229%***	5.065%	5.064%	5.067%	5.070%	5.059%	5.787%	***5.056%***
CC	1.848%	1.849%	1.849%	1.849%	1.848%	***1.846%***	4.157%	4.159%	4.160%	4.161%	4.157%	***4.145%***	4.154%
KC	1.768%	1.773%	1.773%	1.768%	1.768%	***1.768%***	3.982%	3.994%	3.993%	3.982%	3.982%	3.986%	***3.981%***
KW	2.069%	2.068%	2.069%	2.069%	2.066%	***2.063%***	4.664%	4.661%	4.664%	4.664%	4.655%	5.375%	***4.649%***
CT	1.860%	1.864%	1.862%	1.861%	1.859%	***1.859%***	4.197%	4.204%	4.197%	4.198%	4.194%	4.639%	***4.193%***
SB	2.249%	2.256%	2.261%	2.255%	***2.246%***	2.246%	4.631%	4.645%	4.656%	4.643%	4.624%	20.847%	***4.624%***
LA	1.672%	1.672%	1.673%	1.671%	1.669%	***1.669%***	5.565%	5.567%	5.568%	5.561%	5.556%	5.871%	***5.555%***
LL	2.786%	2.784%	2.784%	2.787%	2.778%	***2.778%***	9.246%	9.241%	9.238%	9.252%	9.222%	9.360%	***9.219%***
LP	2.296%	2.300%	2.306%	2.311%	***2.286%***	2.288%	7.621%	7.633%	7.652%	7.670%	***7.586%***	8.458%	7.594%
GC	1.400%	1.400%	1.401%	***1.399%***	1.399%	1.399%	3.426%	3.425%	3.427%	***3.422%***	3.423%	3.552%	3.424%
LN	2.860%	2.859%	***2.857%***	2.862%	2.861%	2.858%	9.436%	9.432%	***9.426%***	9.445%	9.439%	9.490%	9.431%
SI	2.590%	2.589%	2.590%	2.592%	2.587%	***2.586%***	5.790%	5.788%	5.790%	5.795%	5.784%	6.198%	***5.781%***
LX	2.589%	***2.588%***	2.594%	2.591%	2.591%	2.588%	8.555%	***8.552%***	8.571%	8.560%	8.559%	8.660%	8.552%
CL	2.142%	2.144%	2.150%	2.141%	2.139%	***2.137%***	7.518%	7.524%	7.547%	7.514%	7.508%	7.741%	***7.501%***
CO	2.035%	2.036%	2.041%	2.037%	2.029%	***2.028%***	7.118%	7.121%	7.142%	7.124%	7.098%	7.377%	***7.093%***
HO	1.958%	1.960%	1.961%	1.957%	1.951%	***1.949%***	6.865%	6.870%	6.875%	6.861%	6.839%	7.267%	***6.832%***
PG	1.470%	1.471%	1.475%	1.475%	***1.468%***	1.469%	5.112%	5.112%	5.127%	5.128%	***5.103%***	7.807%	5.108%
NG	2.405%	2.403%	2.404%	2.406%	2.401%	***2.396%***	8.814%	8.809%	8.813%	8.818%	8.800%	16.751%	***8.783%***
XB	1.573%	1.573%	1.574%	***1.571%***	1.573%	1.572%	5.296%	5.297%	5.301%	***5.291%***	5.297%	7.111%	5.294%

Test Period

Symbol	Functional Error						Multivariate Error						
	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	Naïve	FAR(1)	FAR(2)	FAR(4)	FSAR	PF	DL	Naïve
FC	1.016%	1.017%	1.016%	1.018%	***1.003%***	1.005%	2.701%	2.703%	2.700%	2.706%	***2.668%***	2.951%	2.672%
LC	0.894%	0.894%	0.894%	0.894%	0.893%	***0.892%***	2.229%	2.231%	2.231%	2.231%	2.227%	3.597%	***2.226%***
LH	1.443%	1.443%	1.443%	***1.432%***	1.443%	1.443%	3.971%	3.971%	3.969%	***3.936%***	3.971%	8.063%	3.970%
C	1.476%	1.476%	1.477%	1.478%	1.474%	***1.473%***	3.346%	3.346%	3.348%	3.351%	3.341%	4.077%	***3.337%***
S	1.235%	1.239%	1.237%	1.239%	1.235%	***1.234%***	3.359%	3.371%	3.362%	3.366%	3.358%	4.149%	***3.354%***
W	1.528%	1.528%	1.528%	1.532%	1.529%	***1.527%***	3.469%	3.469%	3.468%	3.477%	3.471%	3.531%	***3.466%***
CC	1.458%	1.458%	1.462%	1.458%	1.458%	***1.456%***	3.292%	3.292%	3.301%	3.292%	3.291%	3.291%	***3.288%***
KC	1.940%	1.946%	1.946%	1.940%	1.938%	***1.937%***	4.369%	4.384%	4.384%	4.371%	4.365%	***4.351%***	4.363%
KW	1.481%	1.481%	1.482%	1.484%	1.480%	***1.479%***	3.350%	3.348%	3.351%	3.356%	3.346%	3.470%	***3.345%***
CT	1.197%	1.198%	1.206%	1.198%	1.196%	***1.195%***	2.736%	2.738%	2.756%	2.738%	2.733%	3.004%	***2.732%***
SB	1.401%	1.401%	1.403%	1.401%	1.401%	***1.397%***	2.917%	2.918%	2.922%	2.918%	2.918%	11.470%	***2.910%***
LA	1.090%	1.090%	1.090%	1.091%	***1.084%***	1.086%	3.628%	3.627%	3.629%	3.631%	***3.608%***	3.644%	3.613%
LL	1.381%	1.382%	1.382%	1.381%	1.381%	***1.378%***	4.584%	4.585%	4.585%	4.583%	4.583%	4.574%	***4.573%***
LP	1.226%	1.226%	1.227%	1.230%	***1.216%***	1.219%	4.065%	4.066%	4.069%	4.079%	***4.032%***	4.058%	4.042%
GC	1.058%	1.058%	1.058%	1.058%	1.058%	***1.056%***	2.587%	2.588%	2.587%	2.588%	2.588%	2.771%	***2.582%***
LN	1.748%	***1.747%***	1.748%	1.748%	1.748%	1.747%	5.788%	***5.783%***	5.787%	5.788%	5.787%	5.856%	5.784%
SI	1.794%	1.794%	1.796%	1.792%	1.794%	***1.791%***	4.008%	4.006%	4.012%	4.002%	4.006%	4.243%	***4.000%***
LX	1.369%	1.368%	1.370%	1.371%	1.368%	***1.367%***	4.546%	4.546%	4.550%	4.556%	4.543%	***4.542%***	4.542%
CL	1.819%	1.819%	1.819%	1.821%	1.805%	***1.805%***	6.374%	6.377%	6.376%	6.383%	6.326%	6.362%	***6.326%***
CO	1.707%	1.707%	1.709%	1.714%	1.698%	***1.697%***	5.996%	5.996%	6.003%	6.022%	5.965%	6.024%	***5.964%***
HO	1.561%	1.561%	1.561%	1.563%	1.553%	***1.551%***	5.479%	5.480%	5.477%	5.486%	5.450%	5.546%	***5.444%***
PG	2.044%	2.044%	2.046%	2.050%	***2.032%***	2.039%	7.165%	7.167%	7.173%	7.188%	***7.126%***	13.812%	7.148%
NG	1.930%	1.930%	1.930%	1.930%	1.925%	***1.924%***	6.965%	6.966%	6.968%	6.968%	6.946%	11.514%	***6.945%***
XB	2.001%	2.001%	1.999%	2.004%	1.993%	***1.993%***	6.762%	6.762%	6.756%	6.771%	6.737%	11.566%	***6.735%***

Note: red bold values indicate that they are smallest in different forecasting methods.

Appendix C Sample Information

Table 11: Data Information

Symbol	Training Period Start	Training Period End	Validation Period Start	Validation Period End	Test Period Start	Test Period End	Total No. of Obs.
FC	2000-01-04	2005-12-14	2005-12-15	2011-10-31	2011-11-01	2017-08-31	4406
LC	2000-01-04	2005-11-28	2005-11-29	2011-10-13	2011-10-14	2017-08-31	4444
LH	2000-01-04	2005-11-30	2005-12-01	2011-10-14	2011-10-17	2017-08-31	4441
C	2000-01-04	2005-11-23	2005-11-25	2011-10-12	2011-10-13	2017-08-31	4447
S	2000-01-04	2005-11-25	2005-11-28	2011-10-13	2011-10-14	2017-08-31	4446
W	2000-01-04	2005-11-23	2005-11-25	2011-10-12	2011-10-13	2017-08-31	4447
CC	2000-01-04	2005-12-05	2005-12-06	2011-10-26	2011-10-27	2017-08-31	4416
KC	2000-01-04	2005-12-06	2005-12-07	2011-10-25	2011-10-26	2017-08-31	4418
KW	2000-01-04	2005-12-21	2005-12-22	2011-10-26	2011-10-27	2017-08-31	4418
CT	2000-01-04	2005-11-21	2005-11-22	2011-10-18	2011-10-19	2017-08-31	4433
SB	2000-01-04	2005-12-05	2005-12-06	2011-10-24	2011-10-25	2017-08-31	4422
LA	2000-01-05	2006-01-17	2006-01-18	2011-11-08	2011-11-09	2017-08-31	4402
LL	2000-01-05	2006-11-30	2006-12-01	2012-04-18	2012-04-19	2017-08-31	4071
LP	2000-01-05	2006-03-07	2006-03-08	2011-12-01	2011-12-02	2017-08-31	4351
GC	2000-01-05	2005-12-02	2005-12-05	2011-10-19	2011-10-20	2017-08-31	4427
LN	2000-01-05	2006-03-13	2006-03-14	2011-12-08	2011-12-09	2017-08-31	4337
SI	2000-01-05	2005-12-06	2005-12-07	2011-10-20	2011-10-21	2017-08-31	4429
LX	2000-01-05	2006-02-27	2006-02-28	2011-12-02	2011-12-05	2017-08-31	4348
CL	2000-01-05	2005-12-02	2005-12-05	2011-10-19	2011-10-20	2017-08-31	4422
CO	2000-01-05	2005-12-22	2005-12-23	2011-11-02	2011-11-03	2017-08-31	4508
HO	2000-01-05	2005-12-06	2005-12-07	2011-10-19	2011-10-20	2017-08-31	4432
PG	2006-04-24	2010-10-06	2010-10-07	2014-03-19	2014-03-20	2017-08-31	2668
NG	2000-01-05	2005-12-05	2005-12-06	2011-10-20	2011-10-21	2017-08-31	4421
XB	2005-11-01	2009-10-13	2009-10-14	2013-09-20	2013-09-23	2017-08-31	2980

Note: The sample period is generally from January 2000 to August 2017. The whole sample is divided into three segments of equal length, namely training period, validation period, and test period. The difference in total number of observations is due to the two issues: 1) the trading of commodity futures operate in different holiday calendars and schedules; 2) we have removed the daily records with missing values. The data availability of PG and XB in Bloomberg starts from April 2006 and November 2005, respectively.

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