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LINEAR GRAPH TRANSFORMATIONS ON SPACES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{H}$ be a Hilbert space of analytic functions with multiplier algebra $\mathcal{M}(\mathcal{H})$, and let

$$\mathcal{M} = \{(f, T_1 f...., T_{n-1} f) : f \in \mathcal{D}\}$$

be an invariant graph subspace for $\mathcal{M}(\mathcal{H})^{(n)}$. Here $n \geq 2$, $\mathcal{D} \subseteq \mathcal{H}$ is a vector-subspace, $T_i : \mathcal{D} \to \mathcal{H}$ are linear transformations that commute with each multiplication operator $M_\varphi \in \mathcal{M}(\mathcal{H})$, and $\mathcal{M}$ is closed in $\mathcal{H}^{(n)}$. In this paper we investigate the existence of nontrivial common invariant subspaces of operator algebras of the type

$$\mathcal{A}_\mathcal{M} = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D} : AT_i f = T_i A f \ \forall f \in \mathcal{D}\}.$$

In particular, for the Bergman space L^2_a we exhibit examples of invariant graph subspaces of fiber dimension 2 such that $\mathcal{A}_\mathcal{M}$ does not have any nontrivial invariant subspaces that are defined by linear relations of the graph transformations for $\mathcal{M}$.

1. Introduction

Let $d \geq 1$, $\Omega \subseteq \mathbb{C}^d$ be an open, connected, and nonempty set, and let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space. If $\varphi \in \mathrm{Hol}(\Omega)$ such that $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$, then φ is called a multiplier and $M_\varphi f = \varphi f$ defines a bounded linear operator on $\mathcal{H}$. We use $\mathcal{M}(\mathcal{H})$ to denote the multiplier algebra of $\mathcal{H}$, $\mathcal{M}(\mathcal{H}) = \{M_\varphi \in \mathcal{B}(\mathcal{H}) : \varphi \text{ is a multiplier}\}$.

A subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a transitive algebra if it contains the identity operator and if it has no nontrivial common invariant subspaces. It is a longstanding open question (due to Kadison), called the transitive algebra problem, to decide whether every transitive algebra is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology. If that were the case, then, as is well-known, it would easily follow that every $T \in \mathcal{B}(\mathcal{H})$

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which is not a scalar multiple of the identity has a nontrivial hyperinvariant subspace (see e.g. [27]). Recall that a subspace $\mathcal{M}$ is called hyperinvariant for an operator A, if it is invariant for every bounded operator that commutes with A.

Arveson was the first to systematically study the transitive algebra problem. We say that an operator A (respectively an algebra $\mathcal{A}$) has the transitive algebra property, if every transitive algebra that contains A (respectively $\mathcal{A}$) is strongly dense in $\mathcal{B}(\mathcal{H})$. Arveson showed that any maximal abelian self-adjoint subalgebra and the unilateral shift have the transitive algebra property. We refer the reader to [27] for further early results on the transitive algebra problem.

Arveson's approach requires a detailed knowledge of the invariant subspace structure of the operator or the algebra that is to be shown to have the transitive algebra property. Thus based on information about the invariant subspaces of the Dirichlet space Richter was able to use Arveson's approach to establish that the Dirichlet shift has the transitive algebra property, [29]. Then more generally Chong, Guo, and Wang, [11], followed a similar strategy to show among other things that $\mathcal{M}(\mathcal{H})$ has the transitive algebra property, whenever $\mathcal{H}$ has a complete Nevanlinna-Pick kernel, i.e. if the reproducing kernel $k_\lambda(z)$ for $\mathcal{H}$ is of the form $k_\lambda(z) = \frac{\overline{f(\lambda)}f(z)}{1-u_\lambda(z)}$, where f is an analytic function and $u_\lambda(z)$ is positive definite and sesquianalytic. This result covers both the unilateral shift and the Dirichlet shift, and without going into further detail we should say that the Chong-Guo-Wang result also covers higher finite multiplicities as well as restrictions to invariant subspaces.

The current paper was motivated by the desire to decide which other multiplier algebras have the transitive algebra property. Although we did not obtain any specific answers, our investigations lead us to consider some interesting questions related to the invariant subspace structure of $\mathcal{M}(\mathcal{H})$. For additional recent work on questions about transitive algebras we refer the reader to [9].

Our starting point is Arveson's Lemma. For its statement we need to define invariant graph subspaces. If $N > 1$ then $\mathcal{H}^{(N)}$ denotes the direct sum of N copies of $\mathcal{H}$, and for an operator $A \in \mathcal{B}(\mathcal{H})$ $A^{(N)}$ is the $N-$ fold ampliation of A, $A^{(N)} : \mathcal{H}^{(N)} \to \mathcal{H}^{(N)}$, $A^{(N)}(x_1, ..., x_N) = (Ax_1, ..., Ax_N)$.

If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is an algebra of bounded operators on $\mathcal{H}$, then a closed subspace $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ is called an invariant graph subspace for $\mathcal{A}$ if there is a linear manifold $\mathcal{D} \subseteq \mathcal{H}$ and linear transformations $T_1, ..., T_{N-1}$:

$\mathcal{D} \to \mathcal{H}$ such that

$$\mathcal{M} = \{(x, T_1 x, ..., T_{N-1} x) : x \in \mathcal{D}\} \tag{1.1}$$

and such that $A^{(N)}\mathcal{M} \subseteq \mathcal{M}$ for every $A \in \mathcal{A}$. The transformations $T_1, ..., T_{N-1}$ are called linear graph transformations for $\mathcal{A}$. Note that if a linear manifold $\mathcal{D}$ and linear transformations $T_1, ..., T_{N-1} : \mathcal{D} \to \mathcal{H}$ are given, then (1.1) defines an invariant graph subspace for $\mathcal{A}$, if and only if $\mathcal{M}$ is closed, $A\mathcal{D} \subseteq \mathcal{D}$ for every $A \in \mathcal{A}$, and $AT_i = T_i A$ on $\mathcal{D}$ for each $i = 1, ..., N-1$. Thus the graph transformations for $N = 2$ correspond to the closed linear transformations that commute with $\mathcal{A}$. Arveson's Lemma states that a transitive algebra $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$ if and only if the only linear graph transformations for $\mathcal{A}$ are multiples of the identity operator, [8]. For a proof (and statement) we also refer the reader to [27], Lemma 8.8.

In Section 2 we will explain how the following theorem is a simple consequence of Arveson's Lemma.

Theorem 1.1. *Let $\mathcal{H} \subseteq Hol(\Omega)$ be a reproducing kernel Hilbert space. $\mathcal{M}(\mathcal{H})$ has the transitive algebra property if and only if the following condition is satisfied:*

Whenever $N > 1$ and

$$\mathcal{M} = \{(f, T_1 f, ..., T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$$

is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ such that for each $\alpha = (\alpha_0, ..., \alpha_{N-1}) \in \mathbb{C}^N$, $\alpha \neq (0, ..., 0)$ the linear transformation

$$L_\alpha : \mathcal{D} \to \mathcal{H},\ L_\alpha = \overline{\alpha}_0 I + \sum_{i=1}^{N-1} \overline{\alpha}_i T_i$$

is 1-1 and has dense range,

then

$$\mathcal{A}_\mathcal{M} = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D} : AT_i f = T_i A f\ \forall f \in \mathcal{D}\}$$

has nontrivial invariant subspaces.

Note that it is easy to see that for any invariant graph subspace $\mathcal{M}$ the collection $\mathcal{A}_\mathcal{M}$ is a strongly closed algebra, contains $\mathcal{M}(\mathcal{H})$, and that $\mathcal{M}$ is an invariant graph subspace for $\mathcal{A}_\mathcal{M}$. In fact, $\mathcal{A}_\mathcal{M}$ is the largest algebra that has $\mathcal{M}$ as an invariant graph subspace. It is clear that for any $\alpha \in \mathbb{C}^N$ the closures of $\ker L_\alpha$ and ran L_α are invariant subspaces for $\mathcal{A}_\mathcal{M}$. We will say that $\mathcal{A}_\mathcal{M}$ does not have any nontrivial invariant subspaces that are determined by linear relations of the graph transformations, if for each $\alpha \in \mathbb{C}^n$ we have $\overline{\ker L_\alpha}, \overline{\text{ran } L_\alpha} \in \{(0), \mathcal{H}\}$.

With this terminology one easily checks that the condition in Theorem 1.1 is equivalent to the two conditions

(i) the set $\{I, T_1, ..., T_{N-1}\}$ is linearly independent, and

(ii) $\mathcal{A}_\mathcal{M}$ does not have any nontrivial invariant subspaces that are determined by linear relations of the graph transformations.

At this point we note that $\mathcal{D} = \text{ran } L_\alpha$ for $\alpha = (1, 0, ..., 0)$. Thus condition (ii) implies that $\mathcal{D}$ is dense in $\mathcal{H}$.

A useful invariant in the study of invariant subspaces $\mathcal{M} \subseteq \mathcal{H}^N$ is the fiber dimension of $\mathcal{M}$. It is defined as follows. If $\lambda \in \Omega$, if $N \geq 1$, and if $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ is a subspace, then the fiber of $\mathcal{M}$ at λ is

$$\mathcal{M}_\lambda = \{(f_1(\lambda), ..., f_N(\lambda)) : (f_1, ..., f_N) \in \mathcal{M}\} \subseteq \mathbb{C}^N.$$

The fiber dimension of $\mathcal{M}$ is

$$\text{fd } \mathcal{M} = \sup_{\lambda \in \Omega} \dim \mathcal{M}_\lambda.$$

A simple argument using determinants shows that fd $\mathcal{M} = \dim \mathcal{M}_\lambda$ for all $\lambda \in \Omega \setminus E$, where E is the zero set of some nontrivial analytic function on Ω, see [15], Section 1.

If $\mathcal{M} \subseteq \mathcal{H}^N$ is an invariant graph subspace, then it is easy to see that

$$\mathcal{M}_\lambda^\perp = \{\alpha \in \mathbb{C}^N : k_\lambda \perp \text{ran } L_\alpha\},$$

see Lemma 2.4. Thus, the condition that ran L_α is dense implies that $\mathcal{M}$ has full fiber dimension at each point, i.e. $\mathcal{M}_\lambda = \mathbb{C}^N$ for all $\lambda \in \Omega$ such that $k_\lambda \neq 0$, see the remark after Lemma 2.4. It follows that the invariant graph subspaces $\mathcal{M}$ considered in Theorem 1.1 all have fiber dimension $N > 1$.

We will see that whenever fd $\mathcal{M} > 1$, then $\mathcal{A}_\mathcal{M} \neq \mathcal{B}(\mathcal{H})$, see Proposition 2.2. In particular, we note that any $\mathcal{A}_\mathcal{M}$ as above that is transitive would be a counterexample to the transitive algebra problem.

It turns out that if $\mathcal{H}$ has a complete Nevanlinna-Pick kernel then every nonzero invariant graph subspace of $\mathcal{M}(\mathcal{H})$ has fiber dimension one. Thus the condition of the theorem is trivially satisfied, because there is no invariant graph subspace of $\mathcal{M}(\mathcal{H})$ that satisfies the hypothesis of the condition (see Section 2 and [11]).

This means that it becomes a question of interest to decide for which spaces $\mathcal{H}$ one can construct examples of invariant graph subspaces which satisfy the condition of Theorem 1.1. In Section 3 of the paper we will outline a strategy for constructing such invariant graph subspaces (in the case $N = 2$), and we will discuss what other nontrivial invariant subspaces the algebra $\mathcal{A}_\mathcal{M}$ may have. In Section 4 we will show that this can be carried out for the Bergman space L^2_a.

All of our results can be derived from the following example.

Example 1.2. Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space, let φ, ψ be multipliers such that $\frac{1}{\varphi-\psi}$ is a multiplier, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$.

Then with $\mathcal{D} = \mathcal{N} + \mathcal{L}$ and $T(f+g) = \varphi f + \psi g$ the space $\mathcal{M} = \{(h, Th) : h \in \mathcal{D}\}$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ of fiber dimension 2.

This is easy to check, we have included details in Section 3. We mention that Hadwin, Liu, and Nordgren, [16], Section 4, also have constructed an example of an invariant graph subspace (of the Bergman space) with fiber dimension 2. However, we note that with their approach one will always have a nonzero α such that L_α does not have dense range. Since any approach to constructing such fiber dimension 2 or higher invariant graph subspaces is of interest, we have included some details in Section 3.

Examples of invariant subspaces with $\mathcal{N} \cap \mathcal{L} = (0)$ can be based on zero sets. Recall that a set $E \subseteq \Omega$ is called a zero set for $\mathcal{H}$ if $I(E) = \{f \in \mathcal{H} : f(\lambda) = 0 \ \forall \lambda \in E\} \neq (0)$. Then if $A, B \subseteq \Omega$ are zero sets for $\mathcal{H}$ such that $A \cup B$ is not a zero set for $\mathcal{H}$, one checks that $I(A)$ and $I(B)$ are invariant subspaces with $I(A) \cap I(B) = (0)$. See [22], Theorem 2, for a concrete example of this. For $\mathcal{S} \subseteq \mathcal{H}$ let $Z(\mathcal{S}) = \{\lambda \in \mathbb{D} : f(\lambda) = 0 \ \forall f \in \mathcal{S}\}$. It turns out that if in Example 1.2 $\lambda \in Z(\mathcal{N}) \cup Z(\mathcal{L})$, then $\dim \mathcal{M}_\lambda < 2$. Hence any examples built from zero sets as above will not satisfy the hypothesis of Theorem 1.1.

Theorem 1.3. *Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms,* ran $(M_z - \lambda)$ *is closed for all $|\lambda| < 1$, and* $\dim \mathcal{H}/z\mathcal{H} = 1$. *Let $\varphi, \psi \in H^\infty$ such that $1/(\varphi - \psi) \in H^\infty$ and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be $\mathcal{M}(\mathcal{H})$-invariant subspaces such that*

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ *is dense in* $\mathcal{H}$,
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$,
(iv) the inner-outer factorizations of $\varphi - \lambda$ and $\psi - \lambda$ have no singular inner factor for any $\lambda \in \mathbb{C}$, and
(v) neither φ nor ψ is a constant function,

then $\mathcal{M}$ as in Example 1.2 satisfies the hypothesis of Theorem 1.1.

Note that condition (iv) is satisfied for example, whenever both φ and ψ extend to be analytic in a neighborhood of $\overline{\mathbb{D}}$, but there are many other examples. In Section 4 we will show that for the Bergman space of the unit disc $\mathbb{D}$,

$$L^2_a = \{f \in \ \mathrm{Hol}(\mathbb{D}) : \|f\|^2 = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\}$$

the hypotheses of this Theorem can be achieved. Since it is clear that functions φ and ψ can be chosen as in the theorem, our result is implied by the following, which is of independent interest.

Theorem 1.4. *There are two closed subspaces $\mathcal{N}, \mathcal{L} \subseteq L^2_a$ with which are invariant for $\mathcal{M}(L^2_a)$ and such that*

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ *is dense in* L^2_a, *and*
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$.

It is well-established that the Bergman shift has a complicated invariant subspace structure. Thus the above result may not come as a surprise. One reason for these perceived complications is the existence of invariant subspaces $\mathcal{N} \subseteq L^2_a$ of high index, i.e. with $\dim \mathcal{N} \ominus z\mathcal{N} > 1$, [7], [19], [21]. It is notable that our construction in this paper is independent of the high index phenomenon. Indeed we will exhibit a space $\mathcal{H} \subseteq \ \mathrm{Hol}(\mathbb{D})$ with no invariant subspaces of high index, but still admitting the above type of example (Theorem 5.1).

For the Bergman space it is a result of Horowitz that there are zero sets whose union is not a zero set, [22]. We start with Horowitz's example and apply a result of Korenblum, which shows how to "push" zeros to the boundary $\partial\mathbb{D}$, [24]. Then we show that if this is done often enough one can end up with the required example.

In the constructed examples the algebras $\mathcal{A}_{\mathcal{M}}$ have no nontrivial invariant subspaces that are defined by linear relations of the graph transformations. Can one show that they have others? We will see that for many choices of φ and ψ one or both of the subspaces $\mathcal{N}$ and $\mathcal{L}$ that were used in the construction of the example turn out to be invariant for $\mathcal{A}_{\mathcal{M}}$.

Theorem 1.5. *Let $\mathcal{H} \subseteq \ Hol(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example 1.2.*

If

$$\varphi(\mathbb{D}) \setminus \overline{\psi(\mathbb{D})} \neq \emptyset,$$

then $\mathcal{N}$ is an invariant subspace for $\mathcal{A}_{\mathcal{M}}$.

In particular, $\mathcal{A}_{\mathcal{M}}$ has a non-trivial invariant subspace.

Similarly, if $\psi(\mathbb{D}) \setminus \overline{\varphi(\mathbb{D})} \neq \emptyset$*, then* $\mathcal{L}$ *is invariant for* $\mathcal{A}_{\mathcal{M}}$.

This will be Theorem 3.5. It raises the question whether the distinguished subspaces $\mathcal{N}$ and $\mathcal{L}$ of Example 1.2 are always invariant for $\mathcal{A}_{\mathcal{M}}$, but we will give an example of carefully chosen zero-based invariant subspaces of the Bergman space and H^∞-functions φ and ψ that satisfy the hypothesis of Example 1.2, but such that neither $\mathcal{N}$ nor $\mathcal{L}$ are invariant for $\mathcal{A}_{\mathcal{M}}$ (see Example 3.11).

A simple way to construct functions φ and ψ that satisfy the hypothesis of Example 1.2 and Theorem 1.3, but do not satisfy the hypothesis of Theorem 1.5 is to let φ be an analytic function that takes the unit disc onto an annulus centered at 0 and to take $\psi = e^{2\pi i t}\varphi$ for some $t \in (0,1)$. In the case that t is rational the following theorem implies that $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces.

Theorem 1.6. *Let* $\mathcal{H} \subseteq$ *Hol*(Ω) *be such that* $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ *with equivalence of norms, let* $\varphi, \psi \in H^\infty$ *such that* $\frac{1}{\varphi-\psi} \in H^\infty$*, and let* $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ *be closed nonzero invariant subspaces of* $\mathcal{M}(\mathcal{H})$ *such that* $\mathcal{N} \cap \mathcal{L} = (0)$*. Let* $\mathcal{M}$ *be the invariant graph subspace as in Example 1.2.*

If there is a $u \in$ *Hol*$(\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})})$ *such that* $u \circ \varphi = u \circ \psi$*, then* $\mathcal{A}_{\mathcal{M}}$ *has a non-trivial invariant subspace.*

This will be Theorem 3.6. We have been unable to establish that $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces in the general case where φ is an analytic function that takes the unit disc onto an annulus centered at 0 and $\psi = e^{2\pi i t}\varphi$ for some irrational $t \in (0,1)$.

2. Some general observations about graph transformations

We start this section with a lemma which is just an adaptation of Arveson's lemma for our situation. It implies that it suffices to investigate algebras of the type $\mathcal{A}_{\mathcal{M}}$.

Lemma 2.1. $\mathcal{M}(\mathcal{H})$ *has the transitive algebra property, if and only if the following condition holds:*

Whenever $\mathcal{M} = \{(x, T_1x, ..., T_{N-1}x) : x \in \mathcal{D}\}$ *is an invariant graph subspace for* $\mathcal{M}(\mathcal{H})$ *such that* $\mathcal{D}$ *is dense in* $\mathcal{H}$ *and at least one of the* T_i*'s is not a multiple of the identity, then* $\mathcal{A}_{\mathcal{M}}$ *has nontrivial invariant subspaces.*

Proof. We start by showing that the condition is sufficient for the transitive algebra property of $\mathcal{M}(\mathcal{H})$. Let $\mathcal{A}$ be a transitive algebra that contains $\mathcal{M}(\mathcal{H})$. We need to show that $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. By Arveson's Lemma it suffices to prove that the only linear graph transformations for $\mathcal{A}$ are multiples of the identity operator, see [27], Lemma 8.8. Thus let $\mathcal{M} = \{(x, T_1x, ..., T_{N-1}x) : x \in \mathcal{D}\}$ be an invariant graph subspace of $\mathcal{A}$ and suppose that there is an $i, 1 \leq i \leq N-1$ such that T_i is not a multiple of the identity. Then clearly $\mathcal{D} \neq (0)$ and since $\mathcal{A}$ is transitive we must have that $\mathcal{D}$ is dense in $\mathcal{H}$. Note that we have $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}}$. Thus $\mathcal{M}$ is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ and hence the hypothesis implies that $\mathcal{A}_{\mathcal{M}}$ is not transitive. But since $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}}$ this would imply that $\mathcal{A}$ is not transitive, a contradiction. Hence all T_i have to be multiples of the identity, and hence $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.

For the converse we suppose that the condition is not satisfied and we will show that $\mathcal{M}(\mathcal{H})$ then does not have the transitive algebra property. Thus our hypothesis now says that there is an invariant graph subspace $\mathcal{M}$ of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{D}$ is dense in $\mathcal{H}$, such that one of the graph transformations is not a multiple of the identity, and such that $\mathcal{A}_{\mathcal{M}}$ is transitive. Since $\mathcal{A}_{\mathcal{M}}$ contains $\mathcal{M}(\mathcal{H})$ it will be the required example, if we show that $\mathcal{A}_{\mathcal{M}}$ is not strongly dense in $\mathcal{B}(\mathcal{H})$. But all the $T_i's$ are linear graph transformations for $\mathcal{A}_{\mathcal{M}}$, so the result follows from the easy direction of Arveson's lemma. ■

The most obvious linear graph transformations are multiplications by meromorphic functions. For $f \in \mathcal{H}$ we let $[f]$ be the smallest $\mathcal{M}(\mathcal{H})$ invariant subspace containing f. Let $f, g \in \mathcal{H}$, $g \neq 0$ and

$$\mathcal{D} = \{h \in [g] : fh/g \in [f]\},$$

then one easily checks that $T = M_{\frac{f}{g}}$ is a closed linear transformation that commutes with M_φ for all $\varphi \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{D}$ contains $\{\varphi g : \varphi \in \mathcal{M}(\mathcal{H})\}$, thus T will be densely defined whenever g is cyclic in $\mathcal{H}$, i.e. whenever $[g] = \mathcal{H}$.

The following Proposition combined with the previous Lemma captures the essence of the known proofs of the fact that the unilateral shift, the Dirichlet shift, and the algebra $\mathcal{M}(\mathcal{H})$ has the transitive algebra property whenever $\mathcal{H}$ has a complete Nevanlinna-Pick kernel, see [8, 11, 27, 29].

Proposition 2.2. *Let $N \geq 2$ and*

$$\mathcal{M} = \{(f, T_1f, ..., T_{N-1}f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$$

be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ such that $\mathcal{D} \neq (0)$.

(a) Then $\mathcal{M}$ has fiber dimension one, if and only if every T_i is a multiplication.

(b) If the fiber dimension of $\mathcal{M}$ is one, then either every T_i is a multiple of the identity and $\mathcal{A}_{\mathcal{M}} = \mathcal{B}(\mathcal{H})$ or $\mathcal{A}_{\mathcal{M}}$ has a nontrivial invariant subspace which is defined by a linear relation of the graph transformations.

(c) If the fiber dimension of $\mathcal{M}$ is > 1, then $\mathcal{A}_{\mathcal{M}} \neq \mathcal{B}(\mathcal{H})$.

Proof. (a) Suppose for each i we have $T_i = M_{\varphi_i}$ for some meromorphic function φ_i. Let $f_0 \in \mathcal{D}$ with $f_0 \neq 0$. For $\lambda \in \Omega$ such that $f_0(\lambda) \neq 0$ and λ is not a pole of any of the φ_i set

$$u_\lambda = (f_0(\lambda), \varphi_1(\lambda) f_0(\lambda), ..., \varphi_{N-1}(\lambda) f_0(\lambda)) \in \mathbb{C}^N.$$

Then one easily checks that for any $f \in \mathcal{D}$ we have

$$(f(\lambda), (T_1 f)(\lambda), ..., (T_{N-1} f)(\lambda)) = f(\lambda)/f_0(\lambda) u_\lambda.$$

Hence $\mathcal{M}_\lambda = \mathbb{C} u_\lambda$ and $\dim \mathcal{M}_\lambda = 1$. This is true for all λ in an open subset of Ω, hence the fiber dimension of $\mathcal{M}$ must be one.

Conversely, suppose that $\mathcal{M}$ has fiber dimension one, and let $f_0 \in \mathcal{D}$ with $f_0 \neq 0$. For $i = 1, ..., N-1$ set $\varphi_i = T_i f_0 / f_0$. Then φ_i is meromorphic.

Let S_0 be the set of zeros of f_0 and let $\lambda \in \mathbb{D} \setminus S_0$. Set

$$u_\lambda = (f_0(\lambda), (T_1 f_0)(\lambda), ..., (T_{N-1} f_0)(\lambda)).$$

Then $0 \neq u_\lambda \in \mathcal{M}_\lambda$. Thus the hypothesis implies that $\dim \mathcal{M}_\lambda = 1$, and for each $f \in \mathcal{D}$ there is $c_\lambda \in \mathbb{C}$ such that

$$(f(\lambda), (T_1 f)(\lambda), ..., (T_{N-1} f)(\lambda)) = c_\lambda u_\lambda.$$

Hence $c_\lambda = f(\lambda)/f_0(\lambda)$ and for $i = 1, ..., N-1$ we have

$$(T_i f)(\lambda) = c_\lambda (T_i f_0)(\lambda) = \varphi_i(\lambda) f(\lambda).$$

Since $T_i f \in \mathcal{H}$ for each i we conclude that for every $f \in \mathcal{D}$ the function $\varphi_i f$ extends to be analytic in Ω and that T_i is multiplication by φ_i.

(b) It follows from (a) that each T_i is a multiplication. Let $E = \{\lambda \in \Omega : k_\lambda = 0\}$, where k_λ is the reproducing kernel for $\mathcal{H}$. Since $\mathcal{M} \neq (0)$ it is clear that $\Omega \setminus E$ is a nonempty open set. If one of the T_i is not a multiple of the identity, then $T_i = M_\varphi$ where φ is not constant on $\Omega \setminus E$. Let $\lambda_0 \in \Omega \setminus E$, then $T_i - \varphi(\lambda_0)$ is not identically equal to 0 and $k_{\lambda_0} \perp \operatorname{ran} T_i - \varphi(\lambda_0)$. Thus the closure of $\operatorname{ran} T_i - \varphi(\lambda_0)$ is a nontrivial invariant subspace of $\mathcal{A}_{\mathcal{M}}$. In fact, in our earlier terminology, we would say that $\mathcal{A}_{\mathcal{M}}$ has a nontrivial invariant subspace that is defined by a linear relation of the graph transformations. This proves (b).

(c) If $\mathcal{A}_{\mathcal{M}} = \mathcal{B}(\mathcal{H})$, then $\mathcal{M}$ is an invariant graph subspace of $\mathcal{B}(\mathcal{H})$. It follows that each linear transformation T_i is a multiple of the identity, and this implies that the fiber dimension of $\mathcal{M}$ is one. ∎

Thus Lemma 2.1 and Proposition 2.2 imply the the following Corollary. We note, as we have in the Introduction, that if $\mathcal{H}$ has a complete Nevanlinna-Pick kernel, then $\mathcal{M}(\mathcal{H})$ has no invariant graph subspaces of fiber dimension > 1.

Corollary 2.3. *$\mathcal{M}(\mathcal{H})$ has the transitive algebra property if and only if the following condition holds:*
Whenever $\mathcal{M}$ is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ of fiber dimension > 1, then $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces.

We will now restrict the class of the invariant graph subspaces that need to be checked by excluding the ones where $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces defined by linear relations of the graph transformations.

Lemma 2.4. *Let $\mathcal{M} = \{(f, T_1 f, ..., T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$, and let $\lambda \in \Omega$, then*

$$\mathcal{M}_\lambda^\perp = \{\alpha \in \mathbb{C}^N : k_\lambda \perp \text{ran } L_\alpha\}.$$

Here as before for $\alpha \in \mathbb{C}^N$ we defined $L_\alpha = \overline{\alpha_0} I + \sum_{i=1}^{N-1} \overline{\alpha_i} T_i$.

In particular it follows that if ran L_α is dense in $\mathcal{H}$ for all nonzero $\alpha \in \mathbb{C}^N$, then $\mathcal{M}_\lambda = \mathbb{C}^N$ for all $\lambda \in \Omega$, $k_\lambda \neq 0$. We omit the proof of this elementary lemma.

Lemma 2.5. *Let $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$.*
If $\mathcal{A}_{\mathcal{M}}$ has no nontrivial invariant subspaces defined by linear relations of the graph transformations, then there is a subspace $\mathcal{K} \subseteq \mathbb{C}^N$ such that $\mathcal{M}_\lambda = \mathcal{K}$ for all $\lambda \in \Omega$ with $k_\lambda \neq 0$.

Proof. Suppose that all invariant subspaces of $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$, and let $\lambda_1, \lambda_2 \in \Omega$ such that $k_{\lambda_1}, k_{\lambda_2} \neq 0$. The lemma will follow, if we show that $\mathcal{M}_{\lambda_1} = \mathcal{M}_{\lambda_2}$.

Let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{N-1}) \in \mathcal{M}_{\lambda_1}^\perp$ then by the previous lemma k_{λ_1} is orthogonal to ran L_α. The closure of ran L_α is an invariant subspace of $\mathcal{A}_{\mathcal{M}}$ that is defined by a linear relation of the graph transformations, and it does not equal $\mathcal{H}$ since $k_{\lambda_1} \neq 0$. Hence the hypothesis implies ran $L_\alpha = (0)$. This implies that $L_\alpha = 0$ whenever $\alpha \in \mathcal{M}_{\lambda_1}^\perp$. This means $\alpha \in \mathcal{M}_\lambda^\perp$ and hence $\mathcal{M}_\lambda \subseteq \mathcal{M}_{\lambda_1}$ for all $\lambda \in \Omega$. In particular then $\mathcal{M}_{\lambda_2} \subseteq \mathcal{M}_{\lambda_1}$, and in fact by symmetry we conclude $\mathcal{M}_{\lambda_1} = \mathcal{M}_{\lambda_2}$. ■

Lemma 2.6. *Let $\mathcal{M} = \{(f, T_1 f, ..., T_{N-1} f : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ such that all invariant subspaces of $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$.*

If $\mathcal{M}$ has fiber dimension $1 \le k \le N$, then there are linear graph transformations $S_1, ..., S_{k-1} : \mathcal{D} \to \mathcal{H}$ such that each S_i is a linear combination of I and $T_1, ..., T_{N-1}$ and such that

$$\mathcal{N} = \{(f, S_1 f, ..., S_{k-1} f : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(k)}$$

is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ with $\mathcal{A}_{\mathcal{N}} = \mathcal{A}_{\mathcal{M}}$, and $L_\alpha^{\mathcal{N}} = \overline{\alpha_0} I + \sum_{i=1}^{k-1} \overline{\alpha_i} S_i$ is 1-1 and has dense range for all nonzero $\alpha \in \mathbb{C}^k$.

Proof. The hypothesis and Lemma 2.5 implies that there is a k-dimensional subspace $\mathcal{L} \subseteq \mathbb{C}^N$ such that $\mathcal{M}_\lambda = \mathcal{L}$ for all $\lambda \in \Omega$ with $k_\lambda \neq 0$. Write $T_0 = I$, then as in the proof of Lemma 2.5 we have $\sum_{i=0}^{N-1} \overline{\alpha}_i T_i = 0$ for all $\alpha = (\alpha_0, ..., \alpha_{N-1}) \in \mathcal{L}^\perp$. This implies that $\{I, T_1, ..., T_{N-1}\}$ spans a k-dimensional subspace of the linear transformations $\mathcal{D} \to \mathcal{H}$. Let $\{S_0, ..., S_{k-1}\}$ be a basis for this space. Since the space contains I we may assume that $S_0 = I$. It is now easy to check that

$$\mathcal{N} = \{(f, S_1 f, ..., S_{k-1} f : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(k)}$$

satisfies the conclusion of the lemma. Indeed, it is immediate that $\mathcal{N}$ is a closed invariant graph subspace of $\mathcal{M}(\mathcal{H})$ and that $\mathcal{A}_{\mathcal{M}} = \mathcal{A}_{\mathcal{N}}$.

Next we note that $\mathcal{A}_{\mathcal{N}}$ satisfies that all invariant subspaces if $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$, since any linear combination of I and $S_1, ..., S_{k-1}$ is a linear combination of I and $T_1, ..., T_{N-1}$. Since $I, S_1, ..., S_{k-1}$ are linearly independent we conclude that for each nonzero $\alpha \in \mathbb{C}^k$ $L_\alpha^{\mathcal{N}} \neq 0$. Thus $\ker L_\alpha^{\mathcal{N}} = (0)$ and ran $L_\alpha^{\mathcal{N}}$ is dense. ∎

Theorem 1.1 follows immediately from Corollary 2.3 and Lemma 2.6.

The following Theorem describes another way one can identify invariant subspaces that the algebra $\mathcal{A}_{\mathcal{M}}$ may have. We will apply this Theorem in the next section.

Theorem 2.7. *Let $\mathcal{M}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$, and suppose that there is a non-constant meromorphic function u on Ω and a nonzero linear subspace $\mathcal{D}_1$ such that multiplication by u, $M_u : \mathcal{D}_1 \to \mathcal{H}$ commutes with every $A \in \mathcal{A}_{\mathcal{M}}$, i.e. whenever $A \in \mathcal{A}_{\mathcal{M}}$, then $A\mathcal{D}_1 \subseteq \mathcal{D}_1$ and $AM_u = M_u A$ on $\mathcal{D}_1$.*

Then $\mathcal{A}_{\mathcal{M}}$ has non-trivial invariant subspaces.

Proof. Let $\lambda \in \Omega$ such that λ is not a pole of u and $k_\lambda \neq 0$. Then $k_\lambda \perp (M_u - u(\lambda)I)f$ for every $f \in \mathcal{D}_1$, and hence the closure of $(M_u - u(\lambda)I)\mathcal{D}_1$ is a non-trivial invariant subspace for $\mathcal{A}_{\mathcal{M}}$. ∎

Another way to look at the previous theorem is to note that if $\mathcal{M}_1$ is the closure of $\{(f, uf) : f \in \mathcal{D}_1\}$, then $\mathcal{M}_1$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ with fiber dimension 1 and $\mathcal{A}_{\mathcal{M}} \subseteq \mathcal{A}_{\mathcal{M}_1}$. Thus the existence of non-trivial invariant subspaces follows from Proposition 2.2 (b).

3. The general set-up for examples.

We will now restrict our attention to the case $N = 2$ and $\Omega = \mathbb{D}$, the open unit disc in $\mathbb{C}$.

We start this section with a discussion of the example by Hadwin, Liu, and Nordgren (see [16]). Recall that if $\mathcal{N}$ is an invariant subspace of $\mathcal{M}(\mathcal{H})$ and if $M_z \in \mathcal{M}(\mathcal{H})$, then $\dim \mathcal{N} \ominus z\mathcal{N}$ is called the index of $\mathcal{N}$.

Example 3.1. [16] A densely defined closed linear transformation T that is not a multiplication, but commutes with $\mathcal{M}(\mathcal{H})$. Thus by Proposition 2.2 the invariant graph subspace $\mathcal{M} = \{(f, Tf) : f \in \mathcal{D}\}$ has fiber dimension 2.

This can be modified to apply to more general situations where one has index 2 invariant subspaces.

Let $\mathcal{L}, \mathcal{N}$ be index 1 invariant subspaces of the Bergman space L^2_a such that they are at a positive angle, assume that $\mathcal{N}$ is a zero set based invariant subspace. As was observed by Hedenmalm [19] the existence of such subspaces follows from the work of Seip, [30].

Then $\mathcal{L} \vee \mathcal{N} = \mathcal{L} + \mathcal{N}$. Let $f \in \mathcal{L}$, $f \neq 0$ and let

$$\mathcal{D} = \{h + g : h \in L^2_a, hf \in \mathcal{L}, g \in \mathcal{N}\},$$

then $\mathcal{D}$ contains the polynomials and hence is dense in L^2_a. Note that if $h+g = 0$ with $h \in L^2_a, hf \in \mathcal{L}, g \in \mathcal{N}$, then $hf = -fg \in \mathcal{L} \subseteq L^2_a$. Thus $fg \in \mathcal{N}$, because it has the correct zeros. This implies $hf, fg \in \mathcal{L} \cap \mathcal{N}$, hence $hf = fg = 0$, i.e. $h = g = 0$. This implies that $T : \mathcal{D} \to L^2_a, T(h+g) = hf + g$ is well-defined.

It is closed also: Indeed, if $h_n + g_n \in \mathcal{D}$ such that $h_n + g_n \to u$ and $h_n f + g_n \to v$, then because of the positive angle condition we have $g_n \to v_1 \in \mathcal{N}$ and hence $h_n \to u - v_1$ and $h_n f \to v - v_1$. This implies that$(u - v_1)f = v - v_1 \in \mathcal{L}$, and hence $u = (u - v_1) + v_1 \in \mathcal{D}$ and $Tu = (u - v_1)f + v_1 = v$. Thus we have the invariant graph subspace

$$\mathcal{M} = \{(h + g, hf + g) : h \in L^2_a, hf \in \mathcal{L}, g \in \mathcal{N}\}.$$

We already observed that T is densely defined, but the range of T will not be dense since $T\mathcal{D} \subseteq \mathcal{L} + \mathcal{N}$ which has index 2. Furthermore, for all points λ in the common zero set of $\mathcal{N}$ the space $\mathcal{M}_\lambda$ is only one-dimensional.

Thus $\mathcal{M}$ will not satisfy the condition of Theorem 1.1.

The following is our basic example, which was mentioned in the Introduction.

Example 3.2. Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space, let φ, ψ be multipliers such that $\frac{1}{\varphi-\psi}$ is a multiplier, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$.

Then with $\mathcal{D} = \mathcal{N} + \mathcal{L}$ and $T(f+g) = \varphi f + \psi g$ the space $\mathcal{M} = \{(h, Th) : h \in \mathcal{D}\}$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ of fiber dimension 2.

Clearly T is well-defined, and $M_u \mathcal{D} \subseteq \mathcal{D}$ and $M_u T = T M_u$ for every multiplier u. If $f_n \in \mathcal{L}, g_n \in \mathcal{N}$ such that $f_n + g_n \to u$ and $\varphi f_n + \psi g_n \to v$, then $(\varphi - \psi) g_n \to \varphi u - v$. Hence by the hypothesis on $\varphi - \psi$ we have $g_n \to u_1 = \frac{\varphi u - v}{\varphi - \psi} \in \mathcal{N}$. Then $f_n \to u_2 = u - \frac{\varphi u - v}{\varphi - \psi} \in \mathcal{L}$, and $v = \varphi u_1 + \psi u_2 = T(u_1 + u_2)$. Thus, T is closed and hence we obtain the invariant graph subspace

$$\mathcal{M} = \{(f+g, \varphi f + \psi g) : f \in \mathcal{L}, g \in \mathcal{N}\}.$$

We have $\mathcal{M}_\lambda = \mathbb{C}^2$ whenever $\lambda \in \mathbb{D} \setminus (Z(\mathcal{L}) \cup Z(\mathcal{N}))$. In this case we have $(1, \varphi(\lambda)) \in \mathcal{M}_\lambda$ and $(1, \psi(\lambda)) \in \mathcal{M}_\lambda$. These vectors are linearly independent since the hypothesis implies that $\varphi(\lambda) \neq \psi(\lambda)$ for all $\lambda \in \mathbb{D}$. However, it is clear that the dimension of $\mathcal{M}_\lambda < 2$ at every $\lambda \in Z(\mathcal{L}) \cup Z(\mathcal{N})$. Thus, according to the remark after Lemma 2.4 in order to have an example satisfying the condition of Theorem 1.1 we will at least need that $Z(\mathcal{L}) = Z(\mathcal{N}) = \emptyset$.

If neither φ nor ψ is a constant function, then $\ker(T - \lambda) = (0)$ for all $\lambda \in \mathbb{C}$. Suppose $f \in \mathcal{L}, g \in \mathcal{N}$ such that $(T - \lambda)(f+g) = 0$. Then $(\varphi - \lambda) f = -(\psi - \lambda) g \in \mathcal{L} \cap \mathcal{N}$. Thus $(\varphi - \lambda) f = -(\psi - \lambda) g = 0$, hence $f = g = 0$.

For $\alpha = (\alpha_0, \alpha_1)$ we have $L_\alpha = \alpha_0 I + \alpha_1 T$, this L_α has dense range for all nonzero $\alpha \in \mathbb{C}^2$, if and only if $\mathcal{L} + \mathcal{N}$ and $(\varphi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N}$ are dense in $\mathcal{H}$ for every $\lambda \in \mathbb{C}$.

Thus in order to establish Theorem 1.3 it will suffice to prove the following Proposition.

Proposition 3.3. *Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, and* ran $(M_z - \lambda)$ *is closed for all $|\lambda| < 1$, and* $\dim \mathcal{H}/z\mathcal{H} = 1$. *Let $\varphi, \psi \in H^\infty$ such that $1/(\varphi - \psi) \in H^\infty$ and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be $\mathcal{M}(\mathcal{H})$-invariant subspaces such that*

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ *is dense in* $\mathcal{H}$,
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$, *and*

(iv) the inner-outer factorizations of $\varphi - \lambda$ and $\psi - \lambda$ have no singular inner factor for any $\lambda \in \mathbb{C}$,

then $(\varphi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N}$ is dense in $\mathcal{H}$ for every $\lambda \in \mathbb{C}$.

Before we prove the Proposition we need a Lemma.

Lemma 3.4. *Let $\mathcal{H} \subseteq Hol(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, and* ran $(M_z - \lambda)$ *is closed for all $|\lambda| < 1$, and* $\dim \mathcal{H}/z\mathcal{H} = 1$.

Let $\mathcal{K} \subseteq \mathcal{H}$ be an $\mathcal{M}(\mathcal{H})$-invariant subspace with $Z(\mathcal{K}) = \emptyset$. If there is a Blaschke product B such that $B\mathcal{H} \subseteq \mathcal{K}$, then $\mathcal{K} = \mathcal{H}$.

Proof. The first part of this proof is a minor modification of Proposition 3.6 of [28]. Let $\lambda \in \mathbb{D}$ and let $f \in \mathcal{K}$ with $f(\lambda) = 0$. We claim that $f/(z - \lambda) \in \mathcal{K}$.

First suppose that $B(\lambda) \neq 0$. As in [28] it follows from the hypothesis on $\mathcal{H}$ that $f/(z - \lambda) \in \mathcal{H}$. Hence by hypothesis $Bf/(z - \lambda) \in \mathcal{K}$. Note that $(B - B(\lambda))/(z - \lambda) \in H^\infty$, thus $\frac{B-B(\lambda)}{z-\lambda} f \in \mathcal{K}$ and this implies $B(\lambda)f/(z - \lambda) \in \mathcal{K}$. Since $B(\lambda) \neq 0$ we conclude that $f/(z - \lambda) \in \mathcal{K}$.

If $B(\lambda) = 0$, then let $\lambda_n \in \mathbb{D}$ with $B(\lambda_n) \neq 0$ and $\lambda_n \to \lambda$. By hypothesis there is a $g \in \mathcal{K}$ with $g(\lambda) \neq 0$. Then for each n we have $h_n = f_n - \frac{f}{g}(\lambda_n) g \in \mathcal{K}$ and $h_n(\lambda_n) = 0$. By what we have already shown, it follows that $h_n/(z - \lambda_n) \in \mathcal{K}$ for each n. The hypothesis on $\mathcal{H}$ implies that $M_z - \lambda I$ is bounded below, then $M_z - \lambda_n I$ will be bounded below with a similar constant for large n. That can be used to show that $h_n/(z - \lambda_n) \to f/(z - \lambda)$. Thus $f/(z - \lambda) \in \mathcal{K}$.

In particular, if $f \in \mathcal{H}$, then since $Bf \in \mathcal{K}$ we conclude that $Bf/(z - \lambda) \in \mathcal{K}$ for every $\lambda \in \mathbb{D}$ with $B(\lambda) = 0$. This easily implies that $Bf/B_n \in \mathcal{K}$, where B_n is the finite Blaschke product determined by the first n simple factors of B. As $n \to \infty$ the hypothesis implies that $Bf/B_n \to f$ weakly, hence $f \in \mathcal{K}$. Thus $\mathcal{K} = \mathcal{H}$. ■

Proof of Proposition 3.3. Let $\lambda \in \mathbb{C}$ and write

$$\mathcal{K} = \overline{(\phi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N}}.$$

We must show that $\mathcal{K} = \mathcal{H}$.

Note that if $z_0 \in \mathbb{D}$, then either $\varphi(z_0) \neq \lambda$ or $\psi(z_0) \neq \lambda$. In either case the hypothesis (iii) implies that there is a function $f \in \mathcal{K}$ such that $f(z_0) \neq 0$, i.e. $Z(\mathcal{K}) = \emptyset$.

It follows from the hypothesis (iv) that there exist Blaschke products B_1, B_2 and bounded outer functions f_1, f_2 such that $\varphi - \lambda = B_1 f_1$ and $\psi - \lambda = B_2 f_2$. Then

$$\mathcal{K} \supseteq (\varphi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N} \supseteq B_1 f_1 B_2 f_2 (\mathcal{L} + \mathcal{N}) = Bf(\mathcal{L} + \mathcal{N})$$

for some Blaschke product B and some bounded outer function f. Since f is outer, there exists a sequence of polynomials p_n such that $p_n f \to 1$ in the weak*-topology of H^∞, hence $M_{p_n f} \to I$ in the weak operator topology. Thus combining this observation with hypothesis (ii) we obtain $\mathcal{K} \supseteq \overline{B\mathcal{H}}$. Hence $\mathcal{K} = \mathcal{H}$ follows from Lemma 3.4. ■

Now let $\mathcal{H}, \mathcal{L}, \mathcal{N}, \varphi, \psi$ be as in Proposition 3.3, set $\mathcal{D} = \mathcal{L} + \mathcal{N}$, and let $\|f+g\|_{\mathcal{D}}$ be the graph norm on $\mathcal{D}$,

$$\|f+g\|_{\mathcal{D}}^2 = \|f+g\|^2 + \|\varphi f + \psi g\|^2.$$

Then one easily checks that $\mathcal{L}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{D}$ which satisfy $\mathcal{L} \cap \mathcal{N} = 0$ and $\mathcal{L} + \mathcal{N} = \mathcal{D}$. Thus there is a projection $P \in \mathcal{B}(\mathcal{D})$ with $\operatorname{ran} P = \mathcal{L}$ and $\ker P = \mathcal{N}$. Let $Q = I - P$.

Theorem 3.5. *Let $\mathcal{H} \subseteq Hol(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example 1.2.*

If

$$\varphi(\mathbb{D}) \setminus \overline{\psi(\mathbb{D})} \neq \emptyset,$$

then $\mathcal{N}$ is an invariant subspace for $\mathcal{A}_{\mathcal{M}}$.

In particular, $\mathcal{A}_{\mathcal{M}}$ has a non-trivial invariant subspace.

Similarly, if $\psi(\mathbb{D}) \setminus \overline{\varphi(\mathbb{D})} \neq \emptyset$, then $\mathcal{L}$ is invariant for $\mathcal{A}_{\mathcal{M}}$.

Proof. Let $A \in \mathcal{A}_{\mathcal{M}}$. We will show that $A \in \mathcal{B}(\mathcal{D})$ and $PAQ = 0$.

From the definition of $\mathcal{A}_{\mathcal{M}}$ we have $A\mathcal{D} \subseteq \mathcal{D}$ and

$$\begin{aligned}\|Ah\|_{\mathcal{D}}^2 &= \|Ah\|^2 + \|TAh\|^2 = \|Ah\|^2 + \|ATh\|^2 \\ &\leq \|A\|^2(\|h\|^2 + \|Th\|^2) = \|A\|^2\|h\|_{\mathcal{D}}^2.\end{aligned}$$

Thus $A, PAQ, M_\varphi, M_\psi \in \mathbb{B}(\mathcal{D})$. For $f \in \mathcal{L}$ and $g \in \mathcal{N}$ we have

$$\begin{aligned}PAQM_\psi(f+g) &= PAQ(\psi f + \psi g) = PA\psi g \\ &= PATg = PTAg = PT(P+Q)AQ(f+g) \\ &= PM_\varphi PAQ(f+g) + PM_\psi QAQ(f+g) \\ &= M_\varphi PAQ(f+g)\end{aligned}$$

Thus $PAQM_\psi = M_\varphi PAQ$ and hence $(PAQ)^* M_\varphi^* = M_\psi^* (PAQ)^*$.

The hypothesis implies that there is a $\lambda_0 \in \mathbb{D}$ such that

$$\operatorname{dist}(\varphi(\lambda_0), \psi(\mathbb{D})) > 0.$$

Then by continuity there is an open neighborhood $\mathcal{U}$ of λ_0 in $\mathbb{D}$ and a $\delta > 0$ such that for all $\lambda \in \mathcal{U}$ and all $z \in \mathbb{D}$ we have $|\psi(z) - \varphi(\lambda)| \geq \delta$, hence $M_\psi - \varphi(\lambda)I$ is invertible. This implies $\ker(M_\psi^* - \overline{\varphi(\lambda)}) = (0)$ for all $\lambda \in \mathcal{U}$.

Let $\lambda \in \mathcal{U}$ and let k_λ be the reproducing kernel for $\mathcal{D}$. We have

$$(M_\psi^* - \overline{\varphi(\lambda)})(PAQ)^* k_\lambda = (PAQ)^*(M_\varphi^* - \overline{\varphi(\lambda)})k_\lambda = 0.$$

This implies that $(PAQ)^* k_\lambda = 0$ for all $\lambda \in \mathcal{U}$. Since finite linear combinations of k_λ, $\lambda \in \mathcal{U}$ are dense in $\mathcal{D}$ we obtain $PAQ = 0$.

Thus if $f \in \mathcal{N} \subseteq \mathcal{D}$, then $f = Qf$ and $Af = (P+Q)Af = PAQf + QAf = QAf \in \mathcal{N}$, i.e. $A\mathcal{N} \subseteq \mathcal{N}$. ■

Theorem 3.6. *Let $\mathcal{H} \subseteq Hol(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi-\psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example 1.2.*

If there is a $u \in Hol(\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})})$ such that $u \circ \varphi = u \circ \psi$, then $\mathcal{A}_\mathcal{M}$ has a non-trivial invariant subspace.

Proof. Let $v = u \circ \varphi = u \circ \psi$, then $v \in H^\infty(\mathbb{D})$. We will show that $M_v : \mathcal{D} \to \mathcal{H}$ commutes with $\mathcal{A}_\mathcal{M}$. Then the result will follow from Theorem 2.7. We will use a special property of our example, namely that $T\mathcal{D} \subseteq \mathcal{D}$.

If $\lambda \in \mathbb{C}$, $\lambda \notin \overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$, then $\frac{1}{\varphi-\lambda} f \in \mathcal{N}$ and $\frac{1}{\psi-\lambda} g \in \mathcal{L}$ for all $f \in \mathcal{N}$ and $g \in \mathcal{L}$. Thus one easily checks that $(T-\lambda)^{-1}(f+g) = \frac{1}{\varphi-\lambda} f + \frac{1}{\psi-\lambda} g$ and for every $A \in \mathcal{A}_\mathcal{M}$ we have $A(T-\lambda)^{-1} = (T-\lambda)^{-1}A$. It follows that $r(T)A = Ar(T)$ for every rational function r with poles outside of $\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$. The hypothesis on u implies that there is a sequence of rational functions r_n such that $r_n \to u$ uniformly in a neighborhood of $\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$. Then $r_n \circ \varphi$ and $r_n \circ \psi$ are bounded sequences in H^∞ that converge pointwise to v. Thus for every $f \in \mathcal{N}$ and $g \in \mathcal{L}$ we have $r_n(T)(f+g) = r_n \circ \varphi f + r_n \circ \psi g \to v(f+g)$ weakly. Hence $Ar_n(T)(f+g) \to AM_v(f+g)$ and $r_n(T)A(f+g) \to M_v A(f+g)$ weakly for each $f \in \mathcal{N}$ and $g \in \mathcal{L}$. Thus $M_v A = AM_v$. ■

A simple way to satisfy the hypothesis that $1/(\varphi - \psi)$ is a multiplier is if $\varphi = \psi + c$ for some constant $c \neq 0$. Then for appropriate $\mathcal{H}$ it is easy to see that the hypotheses of both of the previous theorems are satisfied, thus $\mathcal{A}_\mathcal{M}$ has non-trivial invariant subspaces. For the u in the previous theorem we can take $u(z) = e^{\frac{2\pi i}{c} z}$. Thus $\mathcal{A}_\mathcal{M}$ commutes with M_v, where $v(z) = e^{\frac{2\pi i}{c}\varphi(z)}$. Actually in this case one can verify directly that $\mathcal{A}_{\mathcal{M}}$ commutes with M_φ.

$$\begin{aligned} AM_\varphi(f+g) &= AM_\varphi f + AM_\psi g + cAg \\ &= AT(f+g) + cAg = TAf + TAg + cAg \\ &= M_\varphi Af + M_\psi Ag + cAg \\ &= M_\varphi A(f+g). \end{aligned}$$

This implies that $AM_\varphi = M_\varphi A$ on $\mathcal{H}$.

If $\varphi(z) = z$, then under the hypothesis of Theorem 3.5 the relation $AM_z = M_z A$ implies $A \in \mathcal{M}(\mathcal{H})$, hence $\mathcal{A}_{\mathcal{M}} = \mathcal{M}(\mathcal{H})$. Thus it seems worthwhile to point out that it can happen that $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example 3.7. Take $\mathcal{H} = L^2_a$, $\varphi(z) = z^2$, $\psi = \varphi + c$, for $c \neq 0$, and choose the two subspaces $\mathcal{L}$ and $\mathcal{N}$ as above such that they are invariant under $(Uf)(z) = f(-z)$. For example, take two zero sets A and B such that the union is not a zero set and such that they both accumulate only on a small arc near 1. Then let $A' = A \cup (-A)$ and $B' = B \cup (-B)$. It is well-known that the extremal function for $I(A)$ has an analytic continuation across any arc $I \subseteq \partial\mathbb{D}$ that does not contain any accumulation points of A (see [1], also see Section 5 of the current paper for the definition and further results on Bergman extremal functions). Thus, if f_1 is the extremal function for $I(A)$ and f_2 is the extremal function for $I(-A)$, then it follows easily that $f_1 f_2 \in I(A')$. Hence both A' and B' are zero sets for $\mathcal{H}$ and their union is not a zero set. Now set $\mathcal{L} = I(A')$ and $\mathcal{N} = I(B')$.

One verifies easily that in this case $U \in \mathcal{A}_{\mathcal{M}}$, thus $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example 3.8. Let $\varphi \in \mathrm{Hol}(\mathbb{D})$, $t \in \mathbb{R} \setminus \mathbb{Z}$, $\alpha = e^{2\pi i t} \neq 1$ and such that $\varphi(\mathbb{D}) = \{z \in \mathbb{C} : r < |z| < R\}$, and $\psi = \alpha\varphi$. For example, φ could be the composition of an conformal map of the disc onto a vertical strip and the exponential function,

$$\varphi(z) = \exp(i \log \frac{1-z}{1+z}).$$

Then $|\varphi(z) - \psi(z)| = |1-\alpha||\varphi(z)| > c$. Furthermore, we check that for no $\lambda \in \mathbb{C}$ the function $\varphi - \lambda$ can have a singular inner factor. Since φ has an analytic continuation at every point except $+1$ or -1, it is clear that the only possible singular inner factors of $\varphi - \lambda$ are determined by point masses at 1 or -1. If $\varphi - \lambda$ had a singular inner factor at 1, then we would have $\varphi(r) - \lambda \to 0$ as $r \to 1^-$. But $\varphi(r) - \lambda$ does not converge as $r \to 1^-$. Similarly we see that there is no singular inner factor with mass at -1. Thus this provides an example of the situation of Theorem 1.3, and since $\varphi(\mathbb{D}) = \psi(\mathbb{D})$ Theorem 3.5 does not apply.

Theorem 3.6 applies only if $t = \frac{n}{m}$ is rational, $u(z) = z^m$. Thus if t is irrational we don't know of any non-trivial invariant subspaces of $\mathcal{A}_{\mathcal{M}}$.

Question 3.9. *Can one show that $\mathcal{A}_{\mathcal{M}}$ has non-trivial invariant subspaces in the previous example if t is irrational?*

The next example refines Example 3.7 to show that even in the context of Question 3.9 one can get $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example 3.10. Let $\varphi(z) = \exp(i \log \frac{1-z^2}{1+z^2})$, $\psi(z) = \alpha\varphi(z)$ and assume that $f(z) \in \mathcal{N}$ if and only if $f(-z) \in \mathcal{N}$ and $g(z) \in \mathcal{L}$ if and only if $g(-z) \in \mathcal{L}$. One can achieve this as in Example 3.7. By combining the approach of Example 3.7 with the construction of the next section one can also achieve this with the added property that $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$. As in Example 3.7 the operator $Uf(z) = f(-z)$ will be in $\mathcal{A}_{\mathcal{M}}$. Thus, $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

The subspaces $\mathcal{N}$ and $\mathcal{L}$ play a distinguished role in all our examples, and one may wonder whether it is always true that both are invariant for $\mathcal{A}_{\mathcal{M}}$. While we cannot rule this out for irrational values of t in the context of Question 3.9, we will show that this may not be the case for $t = 1/2$. Since we know from Theorem 3.6 that $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces in this case anyway, we will just work with zero set based invariant subspaces.

Example 3.11. We will construct zero set based invariant subspaces $\mathcal{N}$ and $\mathcal{L}$ of L^2_a with $\mathcal{N} \cap \mathcal{L} = (0)$ and a disc automorphism u such that $C_u\mathcal{N} = \mathcal{L}$ and $C_u\mathcal{L} = \mathcal{N}$ and an H^∞-function φ such that $1/\varphi \in H^\infty$ and $C_u\varphi = -\varphi$. Here C_u is the composition operator with symbol u.

Then we set $\psi = -\varphi = C_u\varphi$. As above $|\varphi - \psi| = 2|\varphi|$ is bounded below, thus with $\mathcal{D} = \mathcal{N} + \mathcal{L}$ this provides an example satisfying the hypothesis of Example 1.2. Furthermore, one now easily checks that $C_u\mathcal{D} \subseteq \mathcal{D}$ and $TC_u = C_uT$ on $\mathcal{D}$. Thus $C_u \in \mathcal{A}_{\mathcal{M}}$ and hence $\mathcal{N}, \mathcal{L} \notin \operatorname{Lat} \mathcal{A}_{\mathcal{M}}$.

To get started we recall the definitions of interpolating and sampling sequences of a space $\mathcal{H}$ of analytic functions on $\mathbb{D}$.

For a sequence $\{\lambda_n\}$ of distinct points in $\mathbb{D}$ we define $T : \mathcal{H} \to l^\infty$ by $Tf = \{\frac{f(\lambda_n)}{\|k_{\lambda_n}\|}\}_n$. Then $\{\lambda_n\}$ is called an interpolating sequence for $\mathcal{H}$, if T is a bounded operator from $\mathcal{H}$ into and onto l^2, and $\{\lambda_n\}$ is called a sampling sequence for $\mathcal{H}$, if there is a constant $c > 0$ such that $c\|f\| \leq \|Tf\|_{l^2} \leq \frac{1}{c}\|f\|$ for all $f \in \mathcal{H}$.

Lemma 3.12. *If $\Gamma \subseteq \mathbb{D}$ is a sampling sequence for $\mathcal{H}$, if $\overline{\mathbb{D}} = D_+ \cup D_-$, where D_+ and D_- are closed semi-discs, then*

$$\Gamma_+ = \Gamma \cap D_+$$

is not a zero-sequence for $\mathcal{H}$.

Proof. Suppose that $f \in \mathcal{H}$ is a non-zero function with $f(\lambda) = 0$ for all $\lambda \in \Gamma_+$. Since Γ is a sampling sequence, there must be a $c > 0$ such that

$$c\|pf\|^2 \leq \sum_{\lambda \in \Gamma \cap D_-} \frac{|pf(\lambda)|^2}{\|k_\lambda\|^2} \leq \|p\|^2_{\infty, D_-} \sum_{\lambda \in \Gamma \cap D_-} \frac{|f(\lambda)|^2}{\|k_\lambda\|^2} \leq \frac{1}{c}\|p\|^2_{\infty, D_-}\|f\|^2$$

for all polynomials p. Fix $\lambda_0 \in \mathbb{D} \setminus D_-$ with $f(\lambda_0) \neq 0$. By Runge's theorem we may choose a sequence of polynomials p_n such that p_n converges to 0 uniformly on D_- and $p_n(\lambda_0) \to 1$. Then the inequality above implies that $\|p_n f\| \to 0$. This contradicts $p_n f(\lambda_0) \to f(\lambda_0) \neq 0$. Thus Γ_+ is not a zero set for $\mathcal{H}$. ∎

Now let $S = \{z \in \mathbb{C} : -1 < \text{ Re } z < 1\}$ and let $\mathbb{H}^+$ denote the upper half plane of $\mathbb{C}$. The function $f(z) = ie^{-\frac{i\pi z}{2}}$ is a conformal map from S onto $\mathbb{H}^+$ with $f(0) = i$. We note that f takes $\{z : 0 < \text{ Re } z < 1\}$ onto the first quadrant and $f^{-1} : \mathbb{H}^+ \to S$ takes rays emanating from 0 to vertical lines in S. If we further let $g(z) = i\frac{1+z}{1-z}$ be a conformal map of $\mathbb{D}$ onto $\mathbb{H}^+$, then $h = f^{-1} \circ g$ is a conformal map from $\mathbb{D}$ onto S. The function $\varphi = e^{ih}$ is bounded and bounded below as required for Example 3.11.

For $a > 1$ and $b > 0$ define the lattice

$$\Lambda(a, b) = \{a^m(bn + i) : m, n \in \mathbb{Z}\}$$

of points in $\mathbb{H}^+$, and consider the corresponding set $\Gamma(a, b) = g^{-1}(\Lambda(a, b))$ in $\mathbb{D}$. Theorem 3 on page 168 of [13] states that $\Gamma(a, b)$ is interpolating for $\mathcal{H} = L^2_a$ if $\frac{2\pi}{b \log a} < \frac{1}{2}$ and $\Gamma(a, b)$ is sampling for L^2_a if $\frac{2\pi}{b \log a} > \frac{1}{2}$.

Now set $a = e^{\frac{\pi^2}{2}}$ so that $f(z + i\pi) = af(z)$ for all $z \in S$, and choose b such that $\frac{2\pi}{b \log a^2} < \frac{1}{2} < \frac{2\pi}{b \log a}$. Then $\Gamma(a^2, b)$ is interpolating and $\Gamma(a, b)$ is sampling for L^2_a.

Set $\Lambda_1 = \{a^{2m}(bn + i) : m, n \in \mathbb{Z}, n \geq 0\}$, $\Lambda_2 = \{a^{2m+1}(bn + i) : m, n \in \mathbb{Z}, n \geq 0\}$ and for $j = 1, 2$ set $\Gamma_j = g^{-1}(\Lambda_j)$. Then Γ_1 and Γ_2 are subsets of interpolating sets for L^2_a, hence they both are zero sets for L^2_a. Furthermore, $\Gamma_1 \cup \Gamma_2 = g^{-1}(\{a^m(bn + i) : m, n \in \mathbb{Z}, n \geq 0\})$ and it follows from the choice of a and b and Lemma 3.12 that $\Gamma_1 \cup \Gamma_2$ is not a zero set for L^2_a. Thus, $\mathcal{N} = I(\Gamma_1)$ and $\mathcal{L} = I(\Gamma_2)$ are nontrivial invariant subspaces with $\mathcal{N} \cap \mathcal{L} = (0)$.

For $z \in \mathbb{D}$ set $u(z) = g^{-1}(ag(z))$, then u is a disc automorphism with $u(\Gamma_1) = \Gamma_2$ and $u(\Gamma_2) = \Gamma_1$. This implies that $C_u\mathcal{N} = \mathcal{L}$ and $C_u\mathcal{L} = \mathcal{N}$. Furthermore one checks that $h(u(z)) = h(z) + i\pi$ for all $z \in \mathbb{D}$. Thus $C_u\varphi = -\varphi$ and this concludes the construction for Example 3.11.

4. Two zero free subspaces of the Bergman space with trivial intersection

In this section we will use the theory of Bergman extremal functions. Let $(0) \neq \mathcal{M} \subseteq L^2_a$ be an invariant subspace of (M_z, L^2_a), and let n be the smallest natural number such that there is an $f \in \mathcal{M}$ with $f^{(n)}(0) \neq 0$. Then the extremal function for $\mathcal{M}$ is the unique function $G \in \mathcal{M}$ such that $\|G\| = 1$ and $G^{(n)}(0) = \sup\{\mathrm{Re}\, f^{(n)}(0) : f \in \mathcal{M}, \|f\| \leq 1\}$. It is easy to see that the extremal function G of $\mathcal{M}$ is contained in $\mathcal{M} \ominus z\mathcal{M}$. Furthermore, for the case of invariant subspaces $\mathcal{M}$ with index 1 it was shown in [5] that G contractively divides $\mathcal{M}$ and G generates $\mathcal{M}$, i.e. for all $f \in \mathcal{M}$ we have $f/G \in L^2_a$ with $\|f/G\| \leq \|f\|$ and $[G] = \mathcal{M}$. In the following we will use these facts without giving further references.

Let μ be a positive discrete measure on the unit circle $\mathbb{T}$, given by a sequence of points $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{T}$ with corresponding masses $0 < w_k < \infty$ such that

$$\mu = \sum_{k=1}^{\infty} w_k \delta_{\lambda_k}.$$

We shall refer to $\{\lambda_k\}$ as the *a-support* of μ.

When $\|\mu\| = \sum_k w_k < \infty$, μ is associated with the singular inner function

$$S_\mu(z) = \exp\left(-\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right)$$

and by $I_\mu = [S_\mu]$ we denote the invariant subspace of $L^2_a(\mathbb{D})$ generated by S_μ. For non-finite measures μ we define I_μ instead by

$$I_\mu = \bigcap \{[S_\nu] \,:\, 0 \leq \nu \leq \mu,\, \|\nu\| < \infty\}.$$

We say that μ is *admissible* when $I_\mu \neq \{0\}$. Since singly generated invariant subspaces have index 1, it follows from [28], Theorem 3.16 that I_μ has index one whenever μ is admissible. Thus I_μ is generated by its extremal function. Furthermore, we note that a routine argument with contractive zero divisors shows that the extremal function for I_μ is nonzero in $\mathbb{D}$. In conclusion, I_μ is zero free whenever μ is admissible.

The aim of this section is to prove the following theorem.

Theorem 4.1. *There exist two positive discrete admissible measures μ and ν such that*

(i) $I_\mu \cap I_\nu = \{0\}$, *and*
(ii) $I_\mu + I_\nu$ *is dense in* L^2_a.

We begin by stating the following well-known proposition.

Proposition 4.2. *Suppose $f \in L^2_a$ is zero free. Then*

(i) $\lim_{r\to 1}(1-r^2)\log\frac{1}{|f(r\lambda)|} \geq 0$ *exists for all* $\lambda \in \mathbb{T}$.
(ii) *For $\lambda \in \mathbb{T}$ and $w > 0$, we have that $f \in I_{w\delta_\lambda}$ if and only if* $\lim_{r\to 1}(1-r^2)\log\frac{1}{|f(r\lambda)|} \geq 4w$.

Proof. Let $D_\lambda \subset \mathbb{D}$ be the disc of radius $1/2$ that is tangent to $\mathbb{T}$ at λ and note that $f|_{D_\lambda}$ is in the Smirnov class N^+ of D_λ. Standard arguments of Nevanlinna theory now give the validity of (i). A proof of (ii) appears in [25], Proposition 11. ■

We use Proposition 4.2 to prove the following lemma.

Lemma 4.3. *Let $\mu = \sum_k w_k \delta_{\lambda_k}$ be admissible. If $\lambda \in \mathbb{T} \setminus \{\lambda_k\}$ and $w > 0$, then $I_\mu \not\subseteq I_{w\delta_\lambda}$.*

Proof. Suppose on the contrary that $I_\mu \subset I_{w\delta_\lambda}$. Let ϕ_μ and $\phi_{w\delta_\lambda}$ be the respective extremal functions for I_μ and $I_{w\delta_\lambda}$, so that $\phi_\mu \in [\phi_{w\delta_\lambda}]$. Then $\phi_\mu/\phi_{w\delta_\lambda} \in L^2_a$, $\|\phi_\mu/\phi_{w\delta_\lambda}\|_{L^2_a} \leq 1$, and

$$\frac{\phi_\mu}{\phi_{w\delta_\lambda}}(0) > \phi_\mu(0).$$

We are now going to demonstrate that $\phi_\mu/\phi_{w\delta_\lambda} \in I_\mu$, contradicting the extremality of ϕ_μ.

To this end we first note that we may write down $\phi_{w\delta_1}$ explicitly using the method for proving Formula (15) in [12],

$$\phi_{w\delta_1}(z) = \frac{1+\frac{2w}{1-z}}{(1+2w)^{1/2}} S_{w\delta_1}(z).$$

from which we deduce that for all k

$$\lim_{r\to 1}(1-r^2)\log|\phi_{w\delta_\lambda}(r\lambda_k)| = 0.$$

Hence, by Proposition 4.2,

$$\lim_{r\to 1}(1-r^2)\log\left|\frac{\phi_{w\delta_\lambda}(r\lambda_k)}{\phi_\mu(r\lambda_k)}\right| = \lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi_\mu(r\lambda_k)|} \geq 4\sum_{\lambda_\ell=\lambda_k} w_\ell.$$

Applying Proposition 4.2 once more we obtain $\phi_\mu/\phi_{w\delta_\lambda} \in I_\mu$. ■

To prove Theorem 4.1, we are going to construct two positive discrete measures

$$\mu = \sum_k w_k \delta_{\lambda_k}, \quad \nu = \sum_\ell v_\ell \delta_{\xi_\ell}.$$

with disjoint a-supports, $\{\lambda_k\} \cap \{\xi_\ell\} = \emptyset$, such that μ and ν are admissible, but $\mu + \nu$ is not. Then I_μ and I_ν are two zero-free cyclic subspaces, $Z(I_\mu) = Z(I_\nu) = \emptyset$. In addition μ will be constructed such that there exist $f \in I_\mu$, $f \neq 0$, that continue analytically across a nonempty open subarc of $\mathbb{T}$. Before proceeding with the construction, let us show how Theorem 4.1 is obtained from it.

Proof of Theorem 4.1. The non-admissibility of $\mu + \nu$ is equivalent to the fact that $I_\mu \cap I_\nu = \{0\}$. It remains to prove that $I_\mu + I_\nu$ is dense in L^2_a.

From the existence of a non-zero $f \in I_\mu$ extending analytically across a subarc of $\mathbb{T}$ it follows that $\operatorname{clos}(I_\mu + I_\nu)$ is an index-one invariant subspace of L^2_a, see e.g. Theorems A and C of [6]. Hence $\operatorname{clos}(I_\mu + I_\nu)$ is generated by its extremal function ϕ, which clearly has no zeros in $\mathbb{D}$. Denote by ϕ_μ and ϕ_ν the respective extremal functions for I_μ and I_ν, and let $f = \phi_\mu/\phi$ and $g = \phi_\nu/\phi$, recalling that $f, g \in L^2_a$ [5].

We claim that $f \in I_\mu$. To see this note that

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi_\nu(r\lambda_k)|} = 0, \quad \lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} \geq 0, \quad \forall k \geq 1,$$

by Proposition 4.2 and Lemma 4.3. So for every $k \geq 1$ we have

$$0 \leq \lim_{r\to 1}(1-r^2)\log\frac{1}{|g(r\lambda_k)|} = -\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} \leq 0,$$

whence

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} = 0.$$

Therefore

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|f(r\lambda_k)|} = \lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi_\mu(r\lambda_k)|} \geq 4\sum_{\lambda_\ell=\lambda_k} w_\ell,$$

proving that $f \in I_\mu$, by Proposition 4.2. Similarly one shows that $g \in I_\nu$.

Now let $\{p_n\}_n$ and $\{q_n\}_n$ be two sequences of polynomials such that $p_n\phi_\mu + q_n\phi_\nu \to \phi$ in L^2_a as $n \to \infty$. By the contractive divisor property of ϕ we obtain that $p_n f + q_n g = \frac{p_n\phi_\mu + q_n\phi_\nu}{\phi} \in I_\mu + I_\nu$ is a Cauchy sequence, hence $p_n f + q_n g \to 1$. That is, $I_\mu + I_\nu$ is dense in L^2_a. ∎

4.1. **Construction of μ and ν.** By the proof of Theorem 4.6 of [22], the set

$$\Lambda = \left\{ \alpha_{n,k} = \left(\frac{4}{5}\right)^{1/3^n} e^{i2\pi k/3^n} \ : \ n \geq 3, \ |k| < \frac{3^n}{4} \right\}$$

is a zero set for L^2_a, contained in the right half of the unit disc. By Korenblum's method [24], Theorem 3, of sweeping zeros out to the boundary it follows that the measure

$$\mu_0 = \sum_{\substack{n \geq 3 \\ |k| < \frac{3^n}{4}}} w_n \delta_{e^{i2\pi k/3^n}}$$

is admissible, where $w_n = 2\frac{1-|\alpha_{n,k}|}{1+|\alpha_{n,k}|} \sim \frac{1}{3^n}$.

Lemma 4.4. *I_{μ_0} contains a nonzero function that continues analytically across the open arc $J = \{z \in \mathbb{C} : |z| = 1$ and $Re\ z < 0\} \subseteq \mathbb{T}$.*

In Section 5 we will use that it follows from the lemma and a known argument that the extremal function ϕ for I_{μ_0} continues analytically across J, see the proof of Lemma 3.1 of [2].

Proof. Since the zero set Λ is contained in $\{z \in \mathbb{D} : \ \mathrm{Re}\ z > 0\}$ it is known that the extremal function G for the zero-based invariant subspace $I(\Lambda)$ continues analytically across J, see [1] or [31]. For $\alpha \in \mathbb{D}$ set $b_\alpha(z) = \frac{\overline{\alpha}}{|\alpha|} \frac{\alpha - z}{1 - \overline{\alpha} z}$ and

$$S_\alpha(z) = e^{-2\frac{1-|\alpha|}{1+|\alpha|} \frac{\frac{\alpha}{|\alpha|} + z}{\frac{\alpha}{|\alpha|} - z}}.$$

In [24] Korenblum shows that if $\alpha \in \mathbb{D}$ and if $f \in L^2_a$ satisfies $f(\alpha) = 0$, then $\|\frac{S_\alpha}{b_\alpha} f\| \leq \|f\|$.

An easy calculation shows that if $K \subseteq \mathbb{C}$ is a compact set such that $K \cap [1, \infty) = \emptyset$, then there is a $c > 0$ such that

$$|1 - \frac{\frac{r-z}{1-rz}}{e^{-2\frac{1-r}{1+r}\frac{1+z}{1-z}}}| \leq c(1-r)^2$$

for all $z \in K$ and all $0 \leq r < 1$.

Since Λ is an L^2_a-zero set we have $\sum_{\alpha \in \Lambda} (1 - |\alpha|)^2 < \infty$ (see [22], Corollary 3.6). Thus the above estimate shows that the product

$$P(z) = \prod_{\alpha \in \Lambda} \frac{b_\alpha}{S_\alpha}$$

converges uniformly on each compact subset of $\mathbb{D} \cup \{\ \mathrm{Re}\ z < 0\}$ with $P(z) \neq 0$ for all z with $\mathrm{Re}\ z < 0$. Thus the function $f = G/P$

has an analytic continuation across J. Let $\{P_m\}$ be the sequence of partial products of P, then by iterating Korenblum's inequality we have $\|G/P_m\| \leq \|G\|$, so $G/P_m \to f$ weakly L^2_a and it follows that $f \in I_{\mu_0}$. ■

For a fixed $J \geq 1$, pick angles $\theta_1, \dots, \theta_J$ such that $\frac{\theta_1}{2\pi}, \dots, \frac{\theta_J}{2\pi}$ are linearly independent over the rational numbers. Then the a-supports of $\mu_1, \dots, \mu_J$ are pairwise disjoint, where μ_j is the rotation of μ_0 by the angle θ_j, $1 \leq j \leq J$. We also introduce some further notation;

$$\mu_{N,j} = \sum_{\substack{3 \leq n \leq N \\ |k| < \frac{3^n}{4}}} w_n \delta_{e^{i(2\pi k/3^n + \theta_j)}}, \quad \mu^N = \sum_{j=1}^{J} \mu_{N,j},$$

letting F_N denote the a-support of μ^N. For later reference we note that $\|\mu^N\| \sim JN$.

The remainder of this section is dedicated to showing that $\sum_{j=1}^{J} \mu_j$ is not an admissible measure for a sufficiently large J. The construction of μ and ν is then finished by letting $\mu = \mu_{J_0+1}$ and $\nu = \sum_{j=1}^{J_0} \mu_j$, where $1 \leq J_0 \leq J$ is the largest index for which $\sum_{j=1}^{J_0} \mu_j$ is admissible.

We will need several lemmas and the construction of a family of curves. The first lemma we leave for the reader to verify. For a finite measure v on $\mathbb{T}$, we denote its Poisson integral on the disc by $P[v](z)$, $z \in \mathbb{D}$.

Lemma 4.5. *Let $h(z) = P[\delta_1](z) = \frac{1-|z|^2}{|1-z|^2}$ and define for integers $K \geq 27$*

$$H_K(z) = \sum_{k=0}^{K-1} h(e^{i2\pi k/K} z), \quad z \in \mathbb{D}.$$

Then $H_K(z) = Kh(z^K)$ and there exists a constant $C > 0$, independent of K, such that $H_K(re^{i\theta}) < C$ whenever $1 - r < \theta^2$ and $|\theta| \leq \pi/K$.

Next, associated with the finite sets $F_N \subseteq \mathbb{T}$, we define curves Γ_N on which we have fairly precise estimates for $\log|f|$. Similar curves were used by Korenblum in [23]. The main difference between our curves and Korenblum's is that ours are required to be uniformly C^2-smooth, while the curves of [23] are not even C^1, see Figure 4.1 on page 116 of [17].

Let $h : [0,1] \to \mathbb{R}$ be defined by $h(t) = \frac{1}{2\pi^2} t^2 (1-t)^2$. For $\varepsilon \in (0, 2\pi]$ and $t \in [0, \varepsilon]$ set $r_\varepsilon(t) = 1 - \varepsilon^2 h(\frac{t}{\varepsilon})$. Then $0 < r_\varepsilon(t) \leq 1$ and $|r'_\varepsilon(t)|$ and $|r''_\varepsilon(t)|$ are bounded uniformly for all $\varepsilon \in (0, 2\pi]$ and $t \in [0, \varepsilon]$. Note also that $r'_\varepsilon(0) = r'_\varepsilon(\varepsilon) = 0$ and $r''_\varepsilon(0) = r''_\varepsilon(\varepsilon) = \frac{1}{\pi^2}$.

Now let $\emptyset \neq F \subseteq \mathbb{T}$ be finite and define the closed path $\gamma_F : [0, 2\pi] \to \overline{\mathbb{D}}$ as follows: If $t \in [0, 2\pi]$ is such that $e^{it} \in F$, then set $\gamma_F(t) = e^{it}$. Otherwise $e^{it} \in I$, where I is some complementary arc of F with endpoints e^{it_0} and e^{it_1}. Then we set $\gamma_F(t) = r_{|I|}(t - t_0)e^{it}$, where $|I|$ is the length of I. The curve Γ_F is defined as the range of γ_F. It is clear that $\Gamma_F \subseteq \overline{\mathbb{D}}$ is a Jordan curve such that $\Gamma_F \cap \mathbb{T} = F$. The properties of the functions r_ε imply that each Γ_F is C^2-smooth and there is a $C > 0$ such that $\|\gamma_F''\|_\infty \leq C$ for all finite nonempty sets $F \subseteq \mathbb{T}$. Furthermore one checks that the Jordan region bounded by Γ_1 is contained in the Jordan region bounded by Γ_2 whenever $F_1 \subseteq F_2$, and that we have the estimate

$$\text{(4.1)} \qquad \frac{1}{8\pi^2} \operatorname{dist}\left(\frac{z}{|z|}, F\right)^2 \leq 1 - |z| \leq \frac{1}{2\pi^2} \operatorname{dist}\left(\frac{z}{|z|}, F\right)^2, \quad z \in \Gamma_F,$$

where dist refers to the geodesic distance along $\mathbb{T}$.

Let φ_F be the Riemann map from the Jordan domain bounded by Γ_F to the unit disc that takes 0 to 0. φ_F extends to be a homeomorphism from the closure of the Jordan domain bounded by Γ_F to the closed unit disc, and the additional uniform smoothness of the curves Γ_F implies the following lemma.

Lemma 4.6. *There are constants $c, C > 0$ such that for all finite nonempty sets $F \subseteq \mathbb{T}$ we have $c < |\varphi_F'(z)| < C$ for all $z \in \Gamma_F$. Furthermore, if ω_F denotes harmonic measure at 0 on Γ_F, then $d\omega_F = |\varphi_F'| \frac{|dz|}{2\pi}$ and hence*

$$\frac{c}{2\pi} \int_{\Gamma_F} h(z)|dz| \leq \int_{\Gamma_F} h(z) d\omega_F(z) \leq \frac{C}{2\pi} \int_{\Gamma_F} h(z)|dz|$$

for all nonnegative Borel measurable functions h on Γ_F. Here $|dz|$ denotes arclength measure.

This follows from Theorem 3.5 of [26].

For $N \in \mathbb{N}$ we will write $\Gamma_N = \Gamma_{F_N}$.

Lemma 4.7. *There exists a constant $D > 0$, independent of J such that for every $N \geq 3$ we have $\log |S_{\mu^N}(z)| \geq -DJ$ for $z \in \Gamma_N \cap \mathbb{D}$.*

Proof. For this proof we introduce the set $\widetilde{F}_N \supset F_N$,

$$\widetilde{F}_N = \left\{ e^{i(2\pi k/3^n + \theta_j)} \, : \, 3 \leq n \leq N, \ 1 \leq j \leq J, \ 0 \leq k \leq 3^n - 1 \right\},$$

and let $\widetilde{\Gamma}_N = \Gamma_{\widetilde{F}_N}$ be the curve defined by use of the complementary arcs of $\widetilde{F}_N$.

Fix for the moment n and j. For a point $z = re^{i\theta} \in \widetilde{\Gamma}_N \cap \mathbb{D}$, let k_0 be a minimizer of

$$\min_{0 \le k \le 3^n - 1} \operatorname{dist}\left(e^{i\theta}, e^{i(2\pi k/3^n + \theta_j)}\right),$$

and let $z_0 = ze^{-i(2\pi k_0/3^n + \theta_j)} = re^{i\theta_0}$. Note that $|\theta_0| \le \pi/3^n$ and $1 - r \le \theta_0^2$ by (4.1). Hence, by Lemma 4.5

$$\sum_{|k| < \frac{3^n}{4}} P\left[\delta_{e^{i\left(2\pi k/3^n + \theta_j\right)}}\right](z) < H_{3^n}(z_0) < C. \tag{4.2}$$

Since the domain enclosed by $\widetilde{\Gamma}_N$ contains the domain enclosed by Γ_N, it follows by the maximum principle for harmonic functions that (4.2) holds also for $z \in \Gamma_N \cap \mathbb{D}$. Noting now that

$$\log \frac{1}{|S_{\mu^N}(z)|} = \sum_{\substack{3 \le n \le N,\, |k| < \frac{3^n}{4} \\ 1 \le j \le J}} w_n P\left[\delta_{e^{i\left(2\pi k/3^n + \theta_j\right)}}\right](z),$$

with $w_n \sim 1/3^n$, the lemma follows. ∎

Proof that $\sum_{j=1}^J \mu_j$ is not admissible for sufficiently large J. Suppose that $\sum_{j=1}^J \mu_j$ is admissible. We will now argue that J has to be smaller than a certain universal constant. Fix $N \ge 3$ and note first that the admissability of $\sum_{j=1}^J \mu_j$ implies that there exists an $\eta > 0$, independent of N, such that there exists a polynomial p such that $f = pS_{\mu^N}$ satisfies $\|f\|_{L^2_a} \le 1$ and $|f(0)| \ge \eta$. In what follows there will be several implied constants that are all independent of both N and J.

With $f = pS_{\mu^N}$ as above and ω_N denoting harmonic measure on Γ_N with pole at 0 we write

$$\int_{\Gamma_N} \log|f(z)|\, d\omega_N(z) = \int_{\Gamma_N} \log|p(z)|\, d\omega_N(z) + \int_{\Gamma_N} \log|S_{\mu^N}(z)|\, d\omega_N(z). \tag{4.3}$$

Since $\|f\|_{L^2_a} \le 1$ we find by (4.1) and the estimate $|f(z)| \le (1-|z|^2)^{-1}$ that

$$|f(z)| \le \frac{8\pi^2}{\operatorname{dist}(z/|z|, F_N)^2}.$$

Letting $\{I_h\}$ be the collection of complementary arcs on $\mathbb{T}$ to F_N, we obtain

$$\int_{\Gamma_N} \log|f(z)|\, d\omega_N(z) \lesssim \int_{\Gamma_N} \log \frac{2\pi}{\mathrm{dist}(\frac{z}{|z|}, F_N)}\, |dz| + \log 2 \tag{4.4}$$
$$\lesssim \int_{\mathbb{T}} \log \frac{2\pi}{\mathrm{dist}(w, F_N)}\, |dw| + \log 2$$
$$\sim \sum_h |I_h| \log \frac{2\pi}{|I_h|}$$
$$\lesssim 1 + \log|F_N| \lesssim N + \log J$$

where $|I_h|$ denotes the length of I_h and $|F_N| \leq 3^N J$ the number of points in F_N. We have used the fact that the entropy $\sum_h |I_h| \log \frac{2\pi}{|I_h|}$ for a fixed number of intervals is maximized when all intervals are of equal size.

We also note that

$$\int_{\Gamma_N} \log|p(z)|\, d\omega_N(z) \geq \log|p(0)| = \log|f(0)| + \log \frac{1}{|S_{\mu^N}(0)|} \tag{4.5}$$
$$= \log|f(0)| + \|\mu_N\| \gtrsim \log \eta + NJ,$$

and by Lemma 4.7 that

$$\int_{\Gamma_N} \log|S_{\mu^N}(z)|\, d\omega_N(z) \gtrsim -J. \tag{4.6}$$

Combining (4.3), (4.4), (4.5), and (4.6), we find

$$N + \log J \gtrsim \log \eta + NJ - J.$$

Letting $N \to \infty$ we conclude that J must be smaller than some universal constant A, $J \leq A$. ■

5. Hilbert spaces without invariant subspaces with large index

We will now show that the previous example can be used to show that the same phenomenon as in Theorem 4.1 can happen in spaces of analytic functions that have a simpler invariant subspace lattice than the Bergman space.

Theorem 5.1. *There is a space $\mathcal{H} \subseteq$ Hol$(\mathbb{D})$ such that every invariant graph subspace $\mathcal{M}$ has the property that ind$\mathcal{M} = fd\mathcal{M}$, and such that there are index 1 invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of $(M_z, \mathcal{H})$ such that $\mathcal{M} \cap \mathcal{N} = (0)$ and $\mathcal{M} + \mathcal{N}$ is dense in $\mathcal{H}$.*

Proof. It follows from the construction in the proof of Theorem 4.1 that the measures μ and ν can be chosen in such a way that the union of their a-supports is disjoint from some non-empty closed arc $I \subseteq \mathbb{T}$ (just take I to be a small arc centered at -1 and choose all θ_j to be sufficiently small). Let σ be the measure defined by $d\sigma = \chi_I |dz| + dA|\mathbb{D}$ and consider the space $P^2(\sigma)$, the closure of the polynomials in $L^2(\sigma)$. Then one verifies that $P^2(\sigma)$ is irreducible and clearly every point of $\mathbb{D}$ defines a bounded point evaluation for $P^2(\sigma)$, i.e. $P^2(\sigma)$ is an analytic P^2-space in the sense of [4] and [3]. For such spaces it was shown that every non-empty M_z-invariant subspace has index 1 [4], and in fact, Carlsson, [10] showed that every $M_z^{(N)}$-invariant subspace of $P^2(\mu)^{(N)}$ satisfies that its index equals its fiber dimension. In particular, the index of each invariant graph subspace equals its fiber dimension.

Now recall from the paragraph following the statement of Lemma 4.4 that the L^2_a-extremal functions G^μ and G^ν of I_μ and I_ν continue analytically to a neighborhood of I. Hence one obtains that $G^\mu_r \to G^\mu$ and $G^\nu_r \to G^\nu$ in $P^2(\sigma)$ as $r \to 1$, here $f_r(z) = f(rz)$. Thus, $G^\mu, G^\nu \in P^2(\sigma)$ and $[G^\mu]_{P^2(\sigma)} \subseteq [G^\mu]_{L^2_a} = I_\mu$ and $[G^\nu]_{P^2(\sigma)} \subseteq [G^\nu]_{L^2_a} = I_\nu$ are two zero-free index 1 invariant subspaces of $P^2(\sigma)$ with trivial intersection. It follows that the theorem holds with $\mathcal{M} = [G^\mu]_{P^2(\sigma)}, \mathcal{N} = [G^\nu]_{P^2(\sigma)}$ and $\mathcal{H} =$ the closure of $\mathcal{M} + \mathcal{N}$ in $P^2(\sigma)$. ∎

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