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REAL OPTIONS AND GAME THEORETICAL APPROACHES TO REAL ESTATE DEVELOPMENT PROJECTS: MULTIPLE EQUILIBRIA AND THE IMPLICATIONS OF DIFFERENT TIE-BREAKING RULES

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ABSTRACT.
This paper contributes to a fast growing literature which introduces game theory in the analysis of real option investments in a competitive setting. Specifically, in this paper we focus on the issue of multiple equilibria and on the implications that different equilibrium selections may have for the pricing of real options and for subsequent strategic decisions.

We present some theoretical results of the necessary conditions to have multiple equilibria and we show under which conditions different tie-breaking rules result in different economic decisions. We then present a numerical exercise using the information set obtained on a real estate development in South London. We find that risk aversion reduces option value and this reduction decreases marginally as negative externalities decrease.

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1. INTRODUCTION

This paper contributes to a novel and fast growing literature which introduces game theory in the analysis of real options investments in competitive settings. Specifically, in this paper we focus on the issue of multiple equilibria and on the implications that different equilibrium selections may have for the valuation of real options. We present some theoretical results and we apply our analysis to the valuation of a real estate development in South London.

The application of real options theory to commercial real estate has developed rapidly during the last 15 years and various pricing models have been applied to real estate developments to value embedded real options (see for example Titman (1985), Grenadier (1996), Williams (1991) and Williams (1993)). The combination of game and real options theory has proved to offer useful insights for the valuation of real estate developments in markets where different developers are in competition. As far as the pricing of real options is concerned, the existing contributions have used either the binomial option valuation method of Rubinstein, Cox and Ross (1979) in discrete time (see for example the early contribution of Smith and Ankum (1993)) or an equilibrium approach in continuous time (see for example Grenadier (1996)). In both approaches to the valuation of real options, the introduction of game theoretical settings – in order to model the competition between developers – implies the possibility of multiple equilibria (i.e. multiple optimal investment decisions) and such multiplicity is problematic for the option pricing methods.

In a set-up with continuous time, it may be the case that two developers find optimal to invest at the same time, but in order to value the option to invest it is necessary to have a first mover (the leader) and a follower. Grenadier proposed a simple tie- breaking rule (toss of a coin) to solve this problem and Huisman et al. (2003) have proposed a more sophisticated solution based on the use of mixed strategies.

In a discrete time framework, developers play a simultaneous game (invest or defer) at the beginning of each period and there can be multiple equilibria in which either developer invests while the other defers. This is problematic because in order to value an investment, it is necessary to have single equilibrium outcomes at each node of the game and it is not a priori clear how to select between multiple equilibria.

In this paper we propose three tie-breaking (or equilibrium selection) rules which are standard in game theory (min-max payoff, coin-toss, mixed strategy) and we show how the use of these rules can imply different valuations and economic conclusions. For example, selecting between multiple equilibria with the min-max payoff rather than using a coin-toss rule implies a more pessimistic valuation of the future payoff from deferring and hence it may imply that investing becomes optimal when deferring would instead be optimal under a coin-toss rule (i.e. a min-max payoff brings the investment decision forward because it gives preference to earlier cash flows).

The possibility of multiple equilibria in set-up with discrete time has been mentioned by Trigeorgis (1996) and Marcato and Limentani (2008), but to our knowledge, no other paper has so far investigated different tie-breaking rules and the relative implications. With this modeling exercise, we indirectly introduce investor's preferences for future outcomes that can be read as different risk aversion/propensity levels. Consequently, in our theoretical framework we model and test both market conditions (i.e. levels of competition) and investor's preferences (i.e. tie-breaking rules) identifying the marginal effect that each one of them has on the other.

This paper is organized in a theoretical analysis and an application of the theory to a case study. In section 2 we introduce the game theoretical framework and show the necessary conditions to have multiple equilibria. In section 3 we highlight under which conditions different tie-breaking rules result in different economic conclusions. Finally, we apply the theoretical results to the valuation of a mixed-use development project in South London (section 4) and we report how different tie-breaking rules imply different valuations and investment decisions in section 5. We draw our main conclusions in section 6.

2. Nash Equilibrium and the case of Multiple Equilibria

Game theory is a discipline that studies situations of strategic interaction, i.e. situations (*games*) in which the action of an individual (*player*) affects the utility (*pay-off*) of other individuals and in which individuals (*players*) behave strategically taking this interdependence into account. In game theory, a *solution concept* is a formal rule predicting which strategies will be adopted by players, therefore predicting the result of the game. A *strategy* consists of a rule specifying which actions a player should take given the actions taken by other players. The most commonly used solution concepts are equilibrium concepts. Loosely speaking, an *equilibrium* consists of a strategy profile (one for each player) such that each player should not have any advantage by changing her strategy. The seminal Nash (1950) equilibrium consists in a strategic profile such that each player's strategy is a best response to the strategies chosen by the other players. Given a set of feasible strategies for a player i and the strategies chosen by the other players, a best response is the strategy associated to highest payoff for player i. In order to clarify these concepts, we introduce the following game in which two players decide whether to invest (I) or defer (D).

Game 1: *Payoffs.*

	I	D
I	$2\ ,\ 2$	$4\ ,\ 3$
D	$3\ ,\ 4$	$1\ ,\ 1$

The game matrix describes four possible outcomes, where the first and second number in each cell respectively represents the payoff for the player which moves along the rows (the "row"player) and the payoff for the player which moves along the columns (the "column"player). If both players invest, each of them gets a payoff equal to 2; if the "row"player invests while the "column"player defers the respectively get a payoff equal to 4 and 3; if the "row"player defers while the "column"player invests they respectively get a payoff equal to 3 and 4; if both players defer each of them gets a payoff equal to 1. This is a *symmetric* game, because for both players a specific action I (D) gives the same payoff, given the action of the other player. In other words symmetry implies that the identities of the players can be changed without changing the payoffs associated to the strategies. In Game 1, given that one player invests, for the other player it is optimal to defer, because deferring gives a payoff equal to 3, whereas investing gives a lower payoff equal to 2. Likewise, given that one player defers, the optimal choice of the other player is to invest because investing gives a payoff equal to 4, whereas deferring would give a lower payoff equal to 1. In a Nash equilibrium every player maximizes her own payoff function, given all the other players strategies, hence in Game 1 there exist two Nash equilibria. The two equilibria are respectively one in which the row player invests while the column player defers and a symmetric one in which the row player defers while the column player invests. An intuitive method to find the Nash equilibria of such a 2-players games is to underline the payoffs corresponding to the best responses. Looking at the payoffs of the row player, if the column player plays I the best response payoff is 3, if the column player plays D the best response payoff is 4. The best response payoffs are identical for the column player given the symmetry of the game. As the game matrix shows, underlying the payoffs which correspond to the best responses we can easily identify the two Nash equilibria of the game as those situations in which both players play the best responses to each other.

Game 1: *Nash Equilibria.*

	I	D
I	2 , 2	$\underline{4}$, $\underline{3}$
D	$\underline{3}$, $\underline{4}$	1 , 1

This is not the only possible case of multiple equilibria. Consider the same game with modified payoffs.

Game 2: *Payoffs.*

	I	D
I	$\underline{4}$, $\underline{4}$	2 , 1
D	1 , 2	$\underline{3}$, $\underline{3}$

Looking at the payoffs of the row player, if the column player plays I the best response for the row player is to invest, as investing gives a payoff of 4 while deferring would give a payoff of 1. If the column player plays D the best response for the row player is to defer, as deferring gives a payoff of 3 while investing would give a lower payoff of 2. The best response payoffs are identical for the column player given the symmetry of the game. As the game matrix shows, underlying the payoffs which correspond to best responses we can easily identify the two Nash equilibria of the game, respectively "invest, invest"(I,I) and "defer, defer"(D,D). It is useful to look at a generalized version of the same game.

Game 3: *Generalized Payoffs.*

	I	D
I	a , a	b , c
D	c , b	d , d

Notice that the symmetry of the game implies that for both players a specific action I (D) gives the payoff a (c) given that the other player plays I and it gives the payoff b (d) given that the other player plays D. Assuming that a, b, c, d ar all different payoffs, we can identify two conditions:
(i) $a > c$, $d > b$
and
(ii) $a < c$, $d < b$.

either of which is necessary and sufficient for the existence of multiple equilibria. If $a > c$ and $d > b$, as the matrix game below shows, the two underlined Nash equilibria are I,I and D,D.

Game 3a. *Generalized Nash equilibria if $a > c$ and $d > b$.*

	I	D
I	$\underline{a}$, $\underline{a}$	b , c
D	c , b	$\underline{d}$, $\underline{d}$

If $a < c$, $d < b$, as the matrix game below shows, the two underlined Nash equilibria are I,D and D,I.

Game 3b. *Generalized Nash equilibria if $a < c$, $d < b$.*

	I	D
I	a , a	$\underline{b}$, $\underline{c}$
D	$\underline{c}$, $\underline{b}$	d , d

Each of the two conditions is necessary for for the existence of multiple equilibria as it is immediate that if $a > c$, $d < b$ the unique equilibrium is I,I and that if $a < c$, $d > b$ the unique equilibrium is D,D.

In the case of multiple equilibria there is a natural problem of equilibrium selection.[1] We hereby introduce three very simple selection (i.e. tie-breaking) rules. The main point we want to illustrate is that different tie-breaking rules may imply

[1]The literature on equilibrium selections is quite vast and we can refer the interested readers to the seminal contribution of Harsany and Selten (1988).

sensible differences in valuations and therefore it is very important to know the implications of different decision rules.

Expected payoff rule. In the case of two simultaneous equilibria, this rule implies that the two equilibria have equal probability (i.e. 50% probability each). Intuitively, given that the two equilibria are exactly symmetric, according to this rule the equilibrium effectively taking place is decided by the toss of a fair coin. Going back to the previous example of game 3, if $a < c$, $d < b$, as the matrix game shows, the two Nash equilibria are I,D and D,I. Under the expected pay-off rule, for each player the expected pay-off from taking part in the game is $\frac{b+c}{2}$. In terms of investor's preferences, this rule would imply an average risk aversion as the investor does not place a different weight on the two possible outcomes.

Min-max payoff rule. In the case of two simultaneous equilibria the rule implies that a player who wants to value the payoff from participating in the game, assigns probability one to the equilibrium in which she gets the lowest payoff. This rule is known as min-max because this is the minimum payoff that will be achieved playing rationally (i.e. playing the best response). It is intuitive that such valuation of the game's payoff is more "pessimistic"than the one of the expected payoff rule and could possibly imply more risk aversion. Going back again to game 3b, given $a < c$, $d < b$ and in addition $b > c$ ($b < c$), for each player the expected payoff from participating in the game is equal to c (b).

Max-max payoff rule. In the case of two simultaneous equilibria the rule implies that a player who wants to value the payoff from participating in the game, assigns probability one to the equilibrium in which she gets the highest payoff. This rule is known as max-max because this is the maximum payoff that will be achieved playing rationally (i.e. playing the best response). It is intuitive that such valuation of the game's payoff is the more "optimistic"than the one of the expected payoff rule and could possibly imply a move towards a more aggressive risk taking position. Going back again to game 3b, given $a < c$, $d < b$ and in addition $b > c$ ($b < c$), for each player the expected payoff from participating in the game is equal to b (c).

It is worth specifying that we have decided to focus on those three simple rules because they are intuitive from a valuation's point of view.[2] To move from the min-max to the max-max rule can be intuitively interpreted as a shift from the most risk averse valuation to the least risk averse valuation as the certainty equivalent that an investor would be willing to pay to enter the game decreases. The expected payoff rule, instead, averages the valuations of the two extreme rules with equal weights. Consequently, we could think of other cases for which the min-max and the max-max valuations are averaged with different weights, moving in a continuous linear function from 0% to 100% for the maximum payoff possible, and from 100% to 0% for the minimum payoff possible.

[2]The theory of equilibrium selection is quite vast and more sophisticated –or complicated– rules could in principle be used.

3. Multiple equilibria and different tie-breaking rules in the binomial valuation model

Smit and Ankum (1993) merge game theory with real option analysis in order to model competition between two investors who both have the option to defer a project. We present a simple version of their model including all the necessary features in order to illustrate our contribution. We consider two investors A and B that are potentially equal and therefore can be modeled through a symmetric game. Each investor has a strategy set $X_i = \{I, D\}$ with $i = A, B$, where (I) is the decision to invest in the project, and (D) is the decision to defer the investment. The decision to defer D, according to the real option analysis , can be modeled as a call option C. Smit and Ankum (1993) consider the binomial model of Rubinstein, Cox and Ross (1979) in order to value the option to defer the investment. Following their model, we label with $S_{t,h}$ the underlying value of the asset at time t after h upwards movements along the binomial tree and K_c represents investment costs. Since we only consider the pure strategies to invest versus defer (i.e. I and D are mutually exclusive), the symmetry of the game implies three possible outcomes in each period:

- When both investors A and B invest, the game ends. In this situation we start to note the first novelty that the addition of competition brings: instead of the usual intrinsic value $S_{t,h} - K_C$ each investor gets a payoff equal to $\nu\, S_{t,h} - K_C$, where ν is the proportion of value when both competitors invest.
- When both investors defer, nature (N) moves (we can have either an upward movement u or a downward movement d) along the binomial tree and the game is repeated.
- When one investor (leader) invests first and the other (follower) decides to invest later, i.e. defers. In this case the payoff of the leader is $\theta\, S_{t,h} - K_C$. Subsequently, when the follower starts to invest her payoff will be some proportion of the value $\zeta\, S_{t,h} - K_C$ (please note that in our modelling exercise we assume that he leader will always obtain a greater payoff than the follower because θ is bigger than ζ.

In our numerical examples we set $\theta = \exp^{\nu-1}$ and $\zeta = 2 * \nu - \theta$. This assures that $\theta > \nu > \zeta$, which is a feature of the Cournot-Stackelberg framework. Increasing ν means decreasing the extent of competition as the percentage of the value which can be appropriated by simultaneous investments decreases in ν. For $\nu \leq 1$, the difference between θ and ζ decreases in ν up to the point that $\nu = \theta = \zeta = 1$. Therefore, for $\nu \leq 1$, the parameters describe a competitive market with first mover advantage and where the extent of first mover advantage decreases in the extent of competition (i.e. the difference between the payoffs of leader and follower increases as the level of market competition increases).

In our numerical examples, we also consider the case of $\nu > 1$. Also for this case, we set $\theta = \exp^{\nu-1}$ and $\zeta = 2 * \nu - \theta$ which imply that $\theta > \nu > \zeta$. Given that $\nu > 1$, a simultaneous investment for the two players increases the value of the asset above S_0. We interpret this case as a situation where two development projects which are started at the same time imply positive externalities.[3] Maintaining that $\theta > \nu > \zeta$ implies that there is still a first mover's advantage. In the case of $\nu > 1$, we do not interpret the first mover advantage as deriving from preventive competition; rather, we interpret this market as one where the first mover has still a relative advantage since moving first gives her better freedom of choice (i.e. to chosse the most profitable investment between the available ones). For $\nu > 1$, the difference between θ and ζ increases in ν. Therefore, for $\nu > 1$, the parameters describe a market characterized by positive externalities, first mover advantage and where the extent of first mover's advantage increases in the extent of positive externalities.

More generally, fixing S_0 and setting $\theta = \exp^{\nu-1}$ and $\zeta = 2 * \nu - \theta$ allows us to vary the value of ν continuously and study the effect of changing the extent of the first mover's advantage by comparative statics.

Insert figure 1 here.

[3]Such case is interesting for many real estate projects as the construction of complementary complexes of buildings (e.g. housing block and shopping centre) may result in higher individual values for the benefit obtained by the presence of the other property. if we take a housing block and a shopping centre as an example, we see that houses will be worth more if there is an accessible shopping centre in the area. And on the other hand, a centre is more valuable if there are new houses built in the area as the determine an increase of potential customers for the same shopping centre's radius. This argument is also true in the case of the development of two similar properties. If we consider the construction of two shopping centres with slightly different focus and tenancy mix, they may attract a bigger number of customers because customers may be willing to travel a longer distance should they find two malls not far from each other. This would increase the radius the two shopping centres serve and then augment the retail spending and, along with it rents (through revenue-related rents) and hence capital values.

Figure 2 illustrates a two period version of this game[4].

Insert figure 2 here.

We can give the payoff matrix of the game played at time t in state of nature h.

Game 4. *Generalized game in node t,h.*

	Invest		Defer	
Invest	$\nu\, S_{t,h} - K_C e^{-r(T-t)}$,	$\nu S_{t,h} - K_C e^{-r(T-t)}$	$\theta\, S_{t,h} - K_C e^{-r(T-t)}$,	$C^{post}_{t,h}$
Defer	$C^{post}_{t,h}$,	$\theta\, S_{t,h} - K_C e^{-r(T-t)}$	$C_{t,h}$,	$C_{t,h}$

where:

$$S_{t,h} = S_0\, u^h\, d^{t-h}$$

$$C^{post}_{t,h} = max\left(\zeta\, S_{t,h} - K_C e^{-r(T-t)} , \frac{1}{e^{r(T-t)}}[qC^{post}_{t+1,h+1} + (1-q)C^{post}_{t+1,h}]\right)$$

$$C^i_{t,h} = \frac{1}{e^{r(T-t)}}\left(qE^i_{t+1,h+1} + (1-q)E^i_{t+1,h}\right)$$

$$q = \frac{e^r - d}{u-d}$$

$\forall t = 0, ..., T$, $\forall h = 0, ..., T$ and $i = A, B$.

$E^i_{t,h}$ indicates the Nash equilibrium payoff for player i in the game played at time t and state of nature h. The fact that it is possible to have equilibria I,D or D,I where the two players obtain different payoffs implies that $E^i_{t,h}$ – and recursively $C^i_{t,h}$ – are indexed by the player's identity i.

In order to complete the formulation we remember that at maturity T we have the following values:

$$\begin{aligned} S_{T,h} &= S_0\, u^h\, d^{T-h} \\ C^{post}_{T,h} &= 0 \\ C^i_{T,h} &= 0. \end{aligned}$$

[4]Differently from the rest of the analysis, in the figure investment costs are not discounted and $\zeta = 1-\theta$.

Once the binomial tree with games in each node is constructed, it becomes easy to set a strategic tree for each investor. In order to value the option to defer, it is necessary to assign a value to each game-node. Hence, it is extremely important to address the question of what value should be assigned to each player in the case of multiple equilibria (i.e. both are equilibria and so a rule needs to be identified).[5]

Generally, following the analysis of game 3 in section 2 we have the following two necessary and sufficient conditions for the existence of multiple equilibria at a generic node t, h: either (i) $\theta\, S_{t,h} - K_C e^{r(T-t)} > C_{t,h}$, $C^{post}_{t,h} > \nu\, S_{t,h} - K_C e^{r(T-t)}$ where equilibria are both I,D and D,I or (ii) $\theta\, S_{t,h} - K_C e^{r(T-t)} < C_{t,h}$, $C^{post}_{t,h} < \nu\, S_{t,h} - K_C e^{r(T-t)}$ where equilibria are both I,I and D,D.

In order to illustrate the possibility of multiple equilibria it is useful to analyze the game matrix at maturity. The first reason for doing so is that the binomial valuation model imposes to start at maturity and to move backwards. The second reason is that the condition for multiple equilibria at a generic node t, h cannot say much on the values of the parameters which will imply multiple equilibria, given that $S_{t,h}, C_{t,h}$ and $C^{post}_{t,h}$ are endogenous variables which depend on the parameters and on the equilibria in the future periods. Instead studying the game at maturity T implies that both $C_{T,h}$ and $C^{post}_{T,h}$ are equal to zero and hence we can find precise conditions on the parameters for multiple equilibria. Consider the following payoff matrix for the game at maturity.

Game 5. *Generalized game in node T,h.*

	Invest	Defer
Invest	$\nu\, S_{T,h} - K_C$, $\nu\, S_{T,h} - K_C$	$\theta\, S_{T,h} - K_C$ 0 , 0
Defer	0 , $\theta\, S_{T,h} - K_C$	0 , 0

We have the following proposition.

Proposition 1. *A sufficient condition to have multiple equilibria is* $\theta > \frac{K_C}{S_{T,h}} > \nu$.

Proof. Start from the sufficient condition for multiple equilibria I,D and D,I at maturity T (as illustrated in game 3 in section 2): $\theta S_{T,h} - K_C e^{r(T-t)} > 0$, $\nu S_{T,h} - K_C e^{r(T-t)} < 0$. Together the two inequalities imply the condition in proposition 1. □

[5]Smit and Ankum (1993) ignore this case, Trigeorgis (1996) mentions this possibility but rules it out, Limentani and Marcato (2008) mention this possibility but modify the payoff function in such a way to rule it out.

The economic intuition behind proposition 1 is that given the extent of first mover advantage – measured by θ and ν – the ratio between development cost K_c and investment value $S_{T,h}$ must be large enough so that the best response to an investment of the competitor is to defer, and at the same time small enough so that the best response to a deferral of the competitor is to invest.
Following the analysis of section 2, we can formally introduce the discussed tie-breaking rules from the perspective of time t:[6]

- Expected payoff rule: $E^i_{t+1,h} = \frac{1}{2}\Sigma_i E^i_{t+1,h}$.
- Min-max payoff rule: $E^i_{t+1,h} = \min\limits_{w.r.t.i}\{E^i_{t+1,h}\}$.
- Max-max payoff rule: $E^i_{t+1,h} = \max\limits_{w.r.t.i}\{E^i_{t+1,h}\}$.

In general terms, given a node $t+1, h+1$ in which there are two equilibria, respectively with payoff $E^1_{t+1,h+1}$ and $E^2_{t+1,h+1}$, and a node $t+1, h$ with a unique equilibrium with payoff $E^i_{t+1,h}$, we can identify a necessary condition such that different tie-breaking rules in node $t+1, h+1$ imply different equilibrium choices in node t, h. Considering only the expected payoff versus min-max rule, referring to game 4 we find the following conditions:
(i) under expected utility rule $C_{t,h} > \theta S_{t,h} - K_C e^{-r(T-t)}$ and $C^{post}_{t,h} > \nu S_{t,h} - K_C e^{-r(T-t)}$. This implies that under the expected utility rule the unique equilibrium is D,D. Also notice that under the expected utility rule, $C_{t,h} = \frac{1}{e^{r(T-t)}}\left(q\left(\frac{E^1_{t+1,h+1}+E^2_{t+1,h+1}}{2}\right) + (1-q)E_{t+1,h}\right)$.
(ii) under min-max rule, $C_{t,h} < \theta S_{t,h} - K_C e^{-r(T-t)}$ and $C^{post}_{t,h} > \nu S_{t,h} - K_C e^{-r(T-t)}$. This implies that under the min-max rule the equilibria are D,I and I,D. Also notice that under the min-max rule, $C_{t,h} = \frac{1}{e^{r(T-t)}}\left(q\text{min}\left(E^1_{t+1,h+1}, E^2_{t+1,h+1}\right) + (1-q)E_{t+1,h}\right)$.
The intuition for which this is a necessary condition such that different tie-breaking rules in node $t+1, h+1$ imply different equilibrium choices (and hence different investment/defer economic decisions) in node t, h is as follows: different tie-breaking rules only have an impact on the payoff of cell D,D in game 4; therefore in order to have different equilibria, one needs that under one rule D,D is an equilibrium, while under the other rule this is not the case. Since the payoffs in cell D,D are higher under the expected utility rule than under the min-max rule, in order to have different

[6]Notice that the rules respectively average, minimize and maximize the equilibrium payoffs with respect to the players' identity i because payoffs are symmetric across players.

equilibria it must be the case that D,D is an equilibrium only under the expected utility rule.

Once again, in order to illustrate more precise conditions on the parameters, it is useful to take the game matrix at maturity. Take a node T, h in which the condition of proposition 1 is satisfied and therefore there are two equilibria D,I and I,D. Also assume that at node $T, h-1$ the unique equilibrium is D,D (hence with payoffs $E^i_{T,h-1} = 0$ for each player i). Consider the following matrix game at node $T-1, h-1$ for the case of expected payoff valuation of the payoff at T, h:

Game 6. *Generalized game at node $T-1, h-1$ in the case of expected utility rule at node T, h.*

	Invest	Defer
Invest	$\nu\, S_{T-1,h-1} - K_C e^{-r}$, $\nu\, S_{T-1,h-1} - K_C e^{-r}$	$\theta\, S_{T-1,h-1} - K_C e^{-r}$, 0
Defer	0 , $\theta\, S_{T-1,h-1} - K_C e^{-r}$	$\frac{q}{2}(\theta S_T - K_C)e^{-r}$, $\frac{q}{2}(\theta S_T - K_C)e^{-r}$

In this case the condition to have an equilibrium D,D is that $\frac{q}{2}(\theta S_T - K_C)e^{-r} > \theta S_{T-1,h-1} - K_C e^{-r}$, which can be rewritten as $K_C > (\frac{2}{2-q})(\theta S_{T-1,h-1}e^r - \frac{q}{2}\theta S_T)$. Consider now the following matrix game at node $T-1, h-1$ for the case of min-max valuation of the payoff at T, h:

Game 7. *Generalized game at node $T-1, h-1$ in the case of min-max rule at node T, h.*

	Invest	Defer
Invest	$\nu\, S_{T-1,h-1} - K_C e^{-r}$, $\nu\, S_{T-1,h-1} - K_C e^{-r}$	$\theta\, S_{T-1,h-1} - K_C e^{-r}$, 0
Defer	0 , $\theta\, S_{T-1,h-1} - K_C e^{-r}$	0 , 0

In this case the conditions to have two equilibria (I,D) (D,I) are that (i) $K_C > \nu S_{T-1,h-1}e^r$ and (ii) $\theta S_{T-1,h-1} - K_C e^{-r} > 0$, which can be rewritten as $\nu S_{T-1,h-1}e^r < K_C < \theta S_{T-1,h-1}e^r$.

We have the following proposition.

Proposition 2. *If the condition of proposition 1 is satisfied, a sufficient condition to have that different tie-breaking rules imply different investment decisions is* $\frac{\theta e^r}{u} > \frac{K_c}{S_{T,h}} > \frac{2\theta}{2-q}\left(\theta \frac{e^r}{u} - \frac{q}{2}\right)$.

Proof. Proposition 1 shows that the condition $\theta S_{T,h} > K_C > \nu S_{T,h}$ is sufficient to have multiple equilibria at maturity node T, h. The analysis of games 6 and 7 shows, given multiple equilibria at node T, h, the conditions such that the expected payoff and min-max rules result in different investment decisions (i.e. equilibria) in node $T-1, h-1$. After noticing that condition (i) of game 7 is already satisfied under the condition of proposition 1, the conditions can be rewritten as $\theta S_{T-1,h-1}e^r > K_c > (\frac{2}{2-q})(\theta S_{T-1,h-1}e^r - \frac{q}{2}\theta S_T)$. According to the binomial valuation model $S_{T,h} = uS_{T-1,h-1}$, plugging this in the last inequality and dividing by $S_{T,h}$, it can be rewritten as $\frac{\theta e^r}{u} > \frac{K_c}{S_{T,h}} > \frac{2\theta}{2-q}\left(\theta\frac{e^r}{u} - \frac{q}{2}\right)$. □

The economic intuition behind proposition 2 is that, given the extent of the first mover's advantage – measured by θ and μ –, the ratio between development costs K_c and investment value $S_{T,h}$ must be large enough so that in an expected payoff valuation to defer is a dominant strategy and, simultaneously, small enough so that in a min-max payoff valuation the best response to a competitor's deferral is to invest. It is also important to notice that the min-max valuation of the future is more pessimistic and therefore implies that it is optimal to invest since the competitor defers, whereas the expected payoff valuation of future outcomes is more optimistic and therefore implies that it is always optimal to defer.

In the following section we apply our framework to analyze the valuation of a real estate development in South London. We consider several values of the parameters ν, θ, ζ and p in order to model different types of competition across developers and their risk attitudes towards future outcomes. We will also study the cases where different tie-breaking rules imply different investment decisions.

4. Numerical Exercise

Development Project in South London, United Kingdom. As a numerical example, we use a development project based on a 6 acres land South of London. Planning permissions have been already granted for offices (1,350k sqm), retail space [supermarket (830 sqm) and retail units (680k sqm)], a 500 space car park and a leisure component [restaurants and bar (830k sqm), swimming pool and health club (480k sqm), casino (259 sqm) and night club (400k sqm)]. The site was acquired at the price of $£12.78$ million and all cash flow data is available to us. Since the difference between the annual cost of $£150.000$ to keep the strategic option open, and the annual income generated by a car park managed on the site is marginal, we assume that there is no either cost or income in deferment other than financial costs related to discounting (i.e. the dividend is equal to zero). The local authority wishes to see the site completely developed and consequently has already granted planning permissions for the actual development to be started within the next five years.

Traditional NPV Approach and Real Option Analysis. In this study, we want to compare our game-theoretical real option results with the value obtained through a static NPV (i.e. Net Present Value) approach. The development phase lasts 39 months, which correspond to $m = 13$ periods of 3 months each. We obtain the Net Present Value by discounting the expected cash flows back to period 0 (i.e. $t = 0$) using an appropriate discount rate calculated as weighted average cost of capital (i.e. WACC) k. For both cash flows and WACC, we use the information set provided by the investor. Applying a DCF model, we find the NPV of the project as follows:

$$NPV = \sum_{t=0}^{m} \frac{CF_t}{(1+k_q)^t} \tag{1}$$

where CF_t is the expected free cash flow at time t and k_q is the quarterly WACC. More specifically, the cash flow at time t is computed as follows:

$$CF_t = INC_t - LAND_t - DEV_t$$

where INC_t, $LAND_t$ and DEV_t respectively refer to income, land acquisition costs and development expenses, all at time t. Note that there is no income in all t except when the completed building (i.e. $t = m$) is sold. By contrast, land acquisition costs are null in every period, except in the first one. If we discount the cash flows provided by the investor, we obtain the following static NPV of the project (if we were not owning the land yet) as:

$$NPV_p = £79.93 - £59.19 - £12.78 = £7.26 \text{ million}$$

where $£79.93$ million is the present value of the selling price at completion, $£59.19$ million is the present value of all development costs and $£12.78$ million is the acquisition price of the land. Clearly, according to the NPV rule, the project should have been accepted.

In addition, as the company already possesses the land, the option to defer is a function of the construction outcome only (i.e. it does not refer to the overall project which also include the land value). Therefore the static NPV of the construction phase only (i.e. NPV_t) is obtained by adding the land cost to the static NPV of the project:

$$NPV_{cp} = £79.93 - £56.95 = £22.98 \text{ million} = NPV_p + LAND_0$$

Parameters Estimation. The value of our option directly depends on 5 parameters: initial value of the selling price S_0, strike price K_C, volatility of selling price σ, maturity time T and risk-free rate rfr. According to the available information set, we can easily set 4 out of 5 of these parameters:

$$S_0 = 79.93m \qquad K_C = 76m \qquad T = 5$$
$$rfr = 5\%$$

As in Marcato $etal.$ (2008) and Limentani and Marcato (2008), and knowing that the selling price at completion is computed as a perpetuity of market rents (i.e. $Rent_t$) discounted at the relative cap rate (i.e. cap), we estimate the volatility of our development project (i.e. σ) by applying the theory of uncertainty propagation to the volatilities of the growth rates of market rents (i.e. h) and cap rates (i.e. g) and their correlation from historical time series:

$$\begin{aligned} \sigma^2 &= \left(\frac{\partial f}{\partial g}\right)^2 \sigma_g^2 + \left(\frac{\partial f}{\partial h}\right)^2 \sigma_h^2 + 2\,\frac{\partial f}{\partial g}\,\frac{\partial f}{\partial h}\,\rho_{gh}\,\sigma_g\,\sigma_h \\ &= \frac{1}{(1+h)^2}\,\sigma_g^2 + \frac{(1+g)^2}{(1+h)^4}\,\sigma_h^2 + 2\,\frac{1+g}{(1+h)^3}\,\rho_{gh}\,\sigma_g\,\sigma_h \end{aligned} \tag{2}$$

We use 1981 to 2007 times series data provided by a worldwide real estate brokerage firm - CB Richard Ellis (i.e. CBRE) - which gave us access to their UK Average Cap Rate and Rental Index. The two quarterly measures indicate respectively the cap rate and market rent of hypothetical fully rented properties with standard specifications (i.e. a CBRE valuer is asked to give the rent and cap rate of the hypothetical property identified by certain specific criteria, with each valuer reporting on the same property every quarter). The following parameters are estimated:

$$g = 6.79\% \qquad \sigma_g = 10.11\% \qquad \sigma_h = 7.14\% \qquad \rho_{gh} = -0.03$$

while h is assumed to be equal to zero. The corresponding annual volatility is $\sigma = 12.84\%$.

Since the maturity T is 5 years, we develop the exercise strategy of the option using a 5-step binomial model, corresponding to a strategy model whereby the investor can reset the strategy annually.

Competition and Positive Externalities. In order to value the project we also need to define the values of ν, θ, ζ. Those parameters determine the market structure. ν indicates the percentage of final price that each developer manages to achieve when they invest at the same.

As already explained in section 3, we set $\theta = \exp^{\nu-1}$ and $\zeta = 2*\nu-\theta$, which is compatible with the conditions of a Cournot-Stackelberg framework where $\theta > \nu > \zeta$. We let ν vary from 0.5 to 1.5, where values up to 1.0 refer to a competitive setting with negative externalities (i.e. project values are eroded by competition) and values above 1.0 represent a market with positive externalities (i.e. the project value is increased by the investment made by the competitor because the two goods are complementary and not substitute). Finally the difference between θ and ζ represents the advantage of being the first mover rather than the follower.

5. Numerical Results

Effects of the number of steps. We first analyze the impact of varying the number of steps N which characterizes the binomial model. By construction, increasing the number of steps increases the range of values that the asset can take. Marcato $etal.$ (2008) and Limentani and Marcato (2008) show that in a standard binomial model without game-theory, increasing the number of steps implies that the value of the deferral option converges to the Black and Scholes value. We show that this result also holds in a binomial model with game theory, where increasing the number of steps brings towards convergency.

Insert figure 3 here.

Insert figure 4 here.

Insert figure 5 here.

Insert figure 6 here.

Optimal Strategies. We analyze the outcomes of the games which are played along the binomial tree. Following the analysis of game 4 in section 3, we can identify three possible outcomes:
(i) Whenever the payoff from deferring is higher than the payoff from investing first, deferring is a dominant strategy and D, D is the unique equilibrium.
(ii) Whenever the payoff from deferring is between those from investing first and from simultaneous investment, both D, I and I, D are equilibria.
(iii) Whenever the payoff from deferring when the other player invests is lower than the payoff from simultaneous investment, I, I is the unique equilibrium. In this case, there is a prisoner dilemma if $I, I < D, D$.

We can notice that the payoffs from investing first and from simultaneous investment increase in ν and therefore it is the value of ν to determine which of the three cases takes place. In particular each node t, h along the tree is characterized by threshold values $\nu'_{t,h}, \nu''_{t,h}, \nu'''_{t,h}$ such that:
(i) if $\nu < \nu'_{t,h}$ D, D is the equilibrium in node t, h
(ii) if $\nu'_{t,h} \leq \nu < \nu''_{t,h}$ both D, I and I, D are equilibria in node t, h
(iii) if $\nu''_{t,h} \leq \nu < \nu'''_{t,h}$ I, I is the equilibrium in node t, h, but $I, I < D, D$ (prisoner dilemma)
(iv) if $\nu \geq \nu'''_{t,h}$ I, I is the equilibrium in node t, h and $I, I > D, D$ (no prisoner dilemma).

In our numerical example (with 100 steps) we find that the threshold values for the initial node are $\nu'_{t,h} = 1.1$, $\nu''_{t,h} = 1.12$, $\nu'''_{t,h} = 1.13$.

Summarizing, we find that increasing the extent of competition (i.e. decreasing ν) decreases the likelihood of investing as greater competition implies smaller payoffs for both first movers and simultaneous investments. This result parallels some related results of Grandier (1995) and Smit and Trigeorgis (2001). In those works the extent of competition cannot be changed exogenously as each competitor's share of market demand is endogenously determined within the equilibrium concept. Nevertheless, as in our model, they show the existence of threshold levels of revenues that determine different equilibria. Likewise our result, they find that no player invests if the revenue is lower than a certain threshold, there is a first mover if the revenue is higher than this threshold and both players invest simultaneously if the revenue is above an even higher threshold.[7]

Effect of ν. In this section we study how the value of deferring changes as a function of ν. Figure 7 plots the payoffs from deferring (blue) and from simultaneous investment (red) as functions of ν, given 100 steps and different tie-breaking rules.

Insert figure 7 here.

We notice from the graphs that, irrespectively of the tie-breaking rule, the payoff from deferral increases exponentially as competition decreases. The reason why the payoff from deferral decreases with competition is straightforward. The value of deferral is given by value of the future (equilibrium) investments. Future equilibrium investments can be simultaneous (in which case the revenue is multiplied by ν) or as first mover ((in which case the revenue is multiplied by θ). The value of future investments decreases in competition because both ν and θ decrease with competition.

The aspect that is more subtle is that the exact value of deferral depends on which are the future equilibria: namely on which nodes have simultaneous investment and first mover's investment as equilibrium strategy. The parameter θ increases exponentially in ν. As θ is a component of some of the future equilibrium values, the value of deferral increases exponentially as θ decreases. However, we also highlight another effect: as ν increases, the value of simultaneous investment increases. After a certain threshold, simultaneous investment becomes an equilibrium outcome. When simultaneous investment is the equilibrium outcome in every node, the value of deferral does not depend on θ. Consequently, from this point onwards the value of deferral coincides with the value of simultaneous investment and hence it increases linearly in ν.

Effect of different tie-breaking rules. In order to show the effect of different tie-breaking rules, we focus on three different market structures:

[7]This parallels our result as in our framework the revenue for the investor increases in ν.

1. High competition: $\nu = 0.55$, $\theta = 0.638$, $\zeta = 0.462$
2. Low competition: $\nu = 0.8$, $\theta = 0.819$, $\zeta = 0.781$
3. Positive externalities: $\nu = 1.2$, $\theta = 1.221$, $\zeta = 1.179$

The following table shows the value of deferral for each market structure for the three different tie-breaking rules:

Table 1. *Value of deferral for different ν and different p*

	$p = 0.1$	$p = 0.5$	$p = 1$
$\nu = 0.55$	0.4	0.8	3.1
$\nu = 0.8$	9.8	10.1	11
$\nu = 1.2$	38	38	38

The effect of p decreases as ν increases. Explanation: when ν is low the value of deferral depends on future equilibria where some of those equilibria are D, I I, D. The tie-breaking rule has an effect because it selects between those different equilibrium values. As we have shown in the previous section, as ν increases the number of future equilibria I, I increases relative to the D, I I, D equilibria and therefore the impact of the tie-breaking rule is reduced (as the tie-breaking rule impacts on a smaller number of equilibria). Finally, for $\nu \geq 1.2$ all the equilibria along the tree are of the I, I type and therefore the tie-breaking rule has no impact at all.

On the other hand in the neighborhood of the threshold values of ν the tie-breaking rule has an impact on the equilibrium strategies as it changes the threshold values of ν (being those functions of p). Figure 9 shows that for $\nu = 1.1$, I, D D, I are equilibria for $p \leq 0.77$, but I, I becomes the equilibrium for $p > 0.77$. A more optimistic tie-breaking rule pushes both competitors to invest.

Insert figure 9 here.

6. Conclusions

This paper presents a model built within previous literature on real option pricing and game theory and contributes to shed light upon strategic settings where the presence of multiple equilibria situations require equilibrium selection criteria. Along with presenting some theoretical results, we apply three different tie-breaking rules (i.e. min-max payoff, coin-toss, max-max strategy) to the valuation of a development project in South London using a binomial option valuation model in a discrete time framework.

Our framework allows us to consider different market structures where we combine different levels of market competition and investors' risk aversion. We show how the use of different tie-breaking rules can imply different strategies and hence different valuation and economic outcomes. We find that risk aversion reduces option value (i.e. the value is higher for a max-max than for a min-max strategy) and this reduction decreases marginally as negative externalities (i.e. disincentives to defer) decrease.

These results are economically important because investors with different risk aversions may decide to use different rules (i.e. weighting between deferral and investment payoffs) and then obtain significantly different option values. This result has important strategic implications which may be further studied by introducing asymmetric investor types within the same market and their different speeds of reaction to competitors' decisions.

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APPENDIX: FIGURES

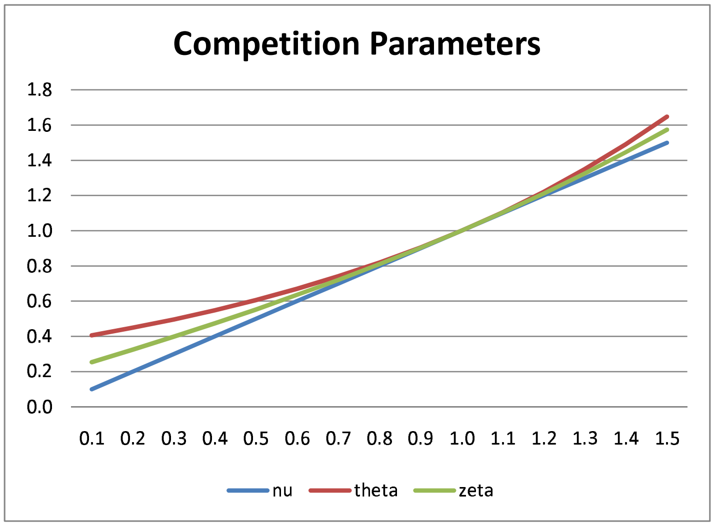


FIGURE 1. Market structure

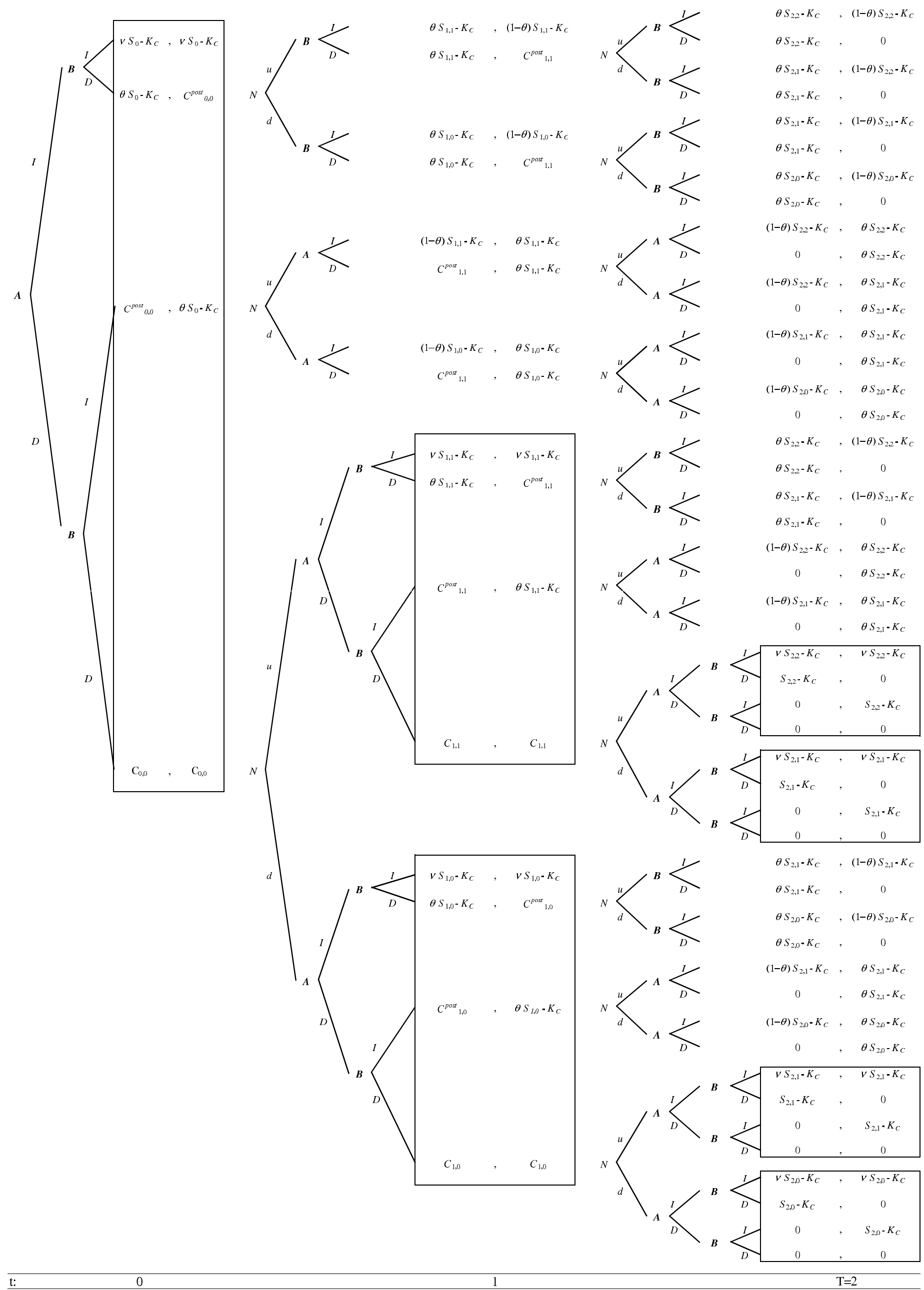
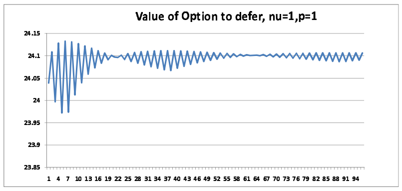
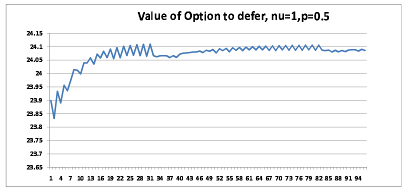


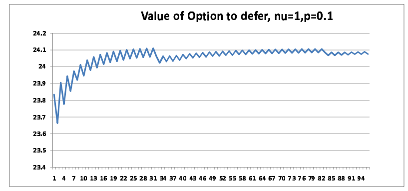
FIGURE 2. Two period strategic game Investment, Defer. Source: Limentani and Marcato (2008).



(A) Deferral option as function of N, for $\nu = 1$, $p = 1$

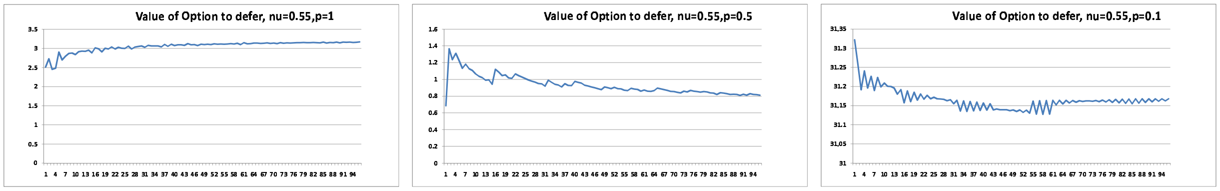


(B) Deferral option as function of N, for $\nu = 1$, $p = 0.5$



(C) Deferral option as function of N, for $\nu = 1$, $p = 0.1$

FIGURE 3. Value of deferral option as a function of N, for $\nu = 1$ and different tie-breaking rules

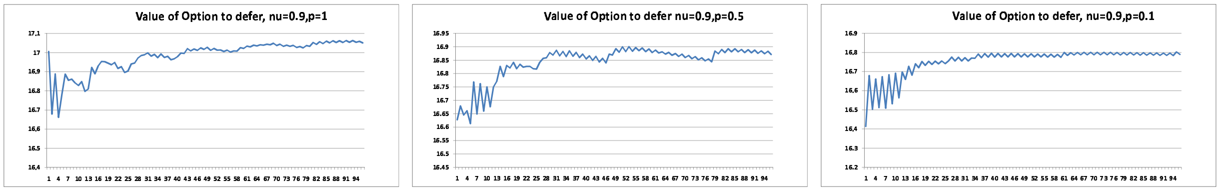


(A) Deferral option as function of N, for $\nu = 0.55$, $p = 1$

(B) Deferral option as function of N, for $\nu = 0.55$, $p = 0.55$

(C) Deferral option as function of N, for $\nu = 0.55$, $p = 0.1$

FIGURE 4. Value of deferral option as a function of N, for $\nu = 0.55$ and different tie-breaking rules

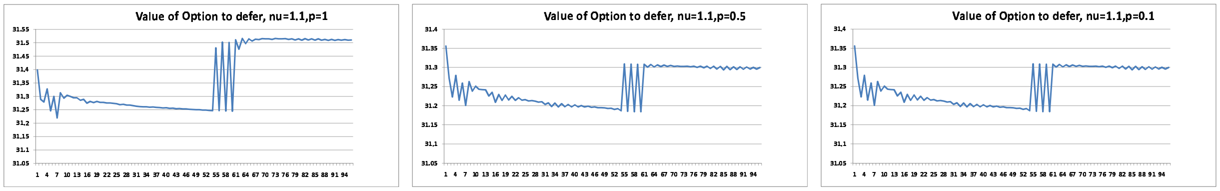


(A) Deferral option as function of N, for $\nu = 0.9$, $p = 1$

(B) Deferral option as function of N, for $\nu = 0.9$, $p = 0.55$

(C) Deferral option as function of N, for $\nu = 0.9$, $p = 0.1$

FIGURE 5. Value of deferral option as a function of N, for $\nu = 0.9$ and different tie-breaking rules

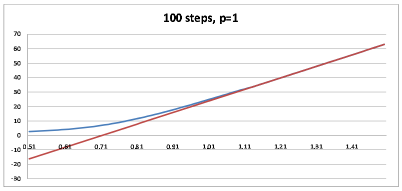


(A) Deferral option (blue) and simultaneous investment (red) as function of N, for $\nu = 1.1$, $p = 1$

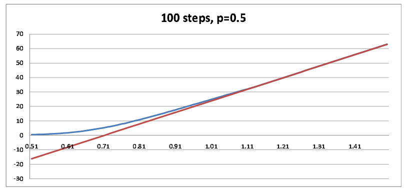
(B) Deferral option (blue) and simultaneous investment (red) as function of N, for $\nu = 1.1$, $p = 0.5$

(C) Deferral option (blue) and simultaneous investment (red) as function of N, for $\nu = 1.1$, $p = 0.1$

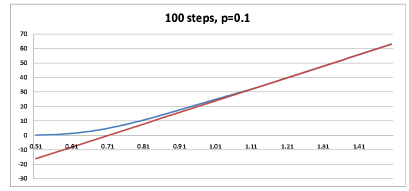
FIGURE 6. Value of deferral option (blue) and simultaneous investment (red) as a function of N, for $\nu = 1.1$ and different tie-breaking rules



(A) Deferral option as function of N, 100 steps and $p = 1$

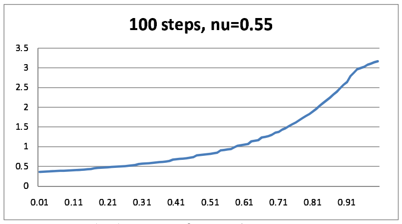


(B) Deferral option as function of N, 100 steps and $p = 0.5$

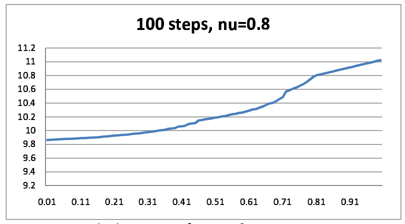


(C) Deferral option as function of N, 100 steps and $p = 0.1$

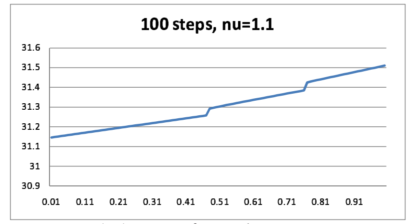
FIGURE 7. Value of deferral option as a function of ν, for 100 steps and different tie-breaking rules



(A) Deferral option as function of p



(B) Deferral option as function of p



(C) Deferral option as function of p

FIGURE 8. Value deferral option for different tie-breaking rules

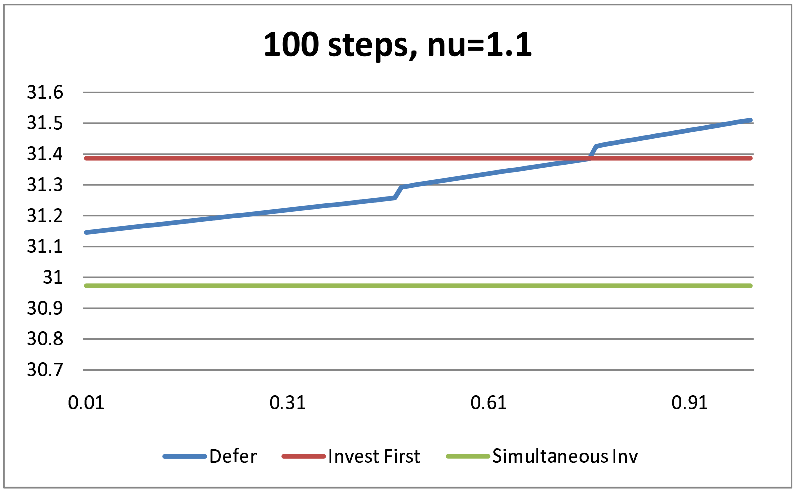


FIGURE 9. Payoffs from different strategies as a function of p, for 100 steps and $\nu = 1.1$