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Multilateral exchange rates: A multivariate regression framework

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Currencies must be priced in terms of a *numéraire* when they are included in a regression model. The numéraire can be either a single-currency numéraire or a multicurrency numéraire: a weighted basket of numéraire currencies. Pricing currencies in terms of a multicurrency numéraire results in a system of multilateral exchange rates. A no-arbitrage condition enforces the movements in the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, where the covariance matrix is singular and its ordinary inverse does not exist. Singular systems pose a methodological challenge in a multivariate regression model. This paper provides a solution to overcome this methodological challenge by imposing implicit restrictions on both the explanatory variables and the regression coefficients. In addition, the generalized least squares estimator is modified by replacing the ordinary inverse with the generalized inverse. The proposed solution provides a consistent multivariate regression model to explain the observed heterogeneity in the relative currency market.

1. Introduction

Currencies are typically priced in terms of a *numéraire*. The numéraire can be either a single-currency numéraire that is a numéraire currency or a multicurrency numéraire that is a basket of numéraire currencies. If a multicurrency numéraire is chosen as the numéraire, each currency in the system of currencies is represented by a multilateral exchange rate. However, a no-arbitrage condition enforces the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, which poses a methodological challenge in a multivariate regression model. The motivation for this paper is to overcome this methodological challenge.

A numéraire must be chosen when currencies are included in both univariate and multivariate regression models. For example, all currencies are priced in terms of a common numéraire for a univariate Frankel-Wei regression model, which is used to measure the relationship between currencies (Frankel & Wei, 1994). The choice for the common numéraire is between a single-currency numéraire or a multicurrency numéraire. If a single-currency numéraire is chosen, the currencies are represented by bilateral exchange rates. As previously mentioned, if a multicurrency numéraire is chosen, the currencies are represented by multilateral exchange rates.

Bilateral exchange rates tend to be used more than multilateral exchange rates (Haynes & Stone, 1994). However, multilateral exchange rates have recently been shown to be superior to bilateral exchange rates in the univariate Frankel-Wei regression framework, as bilateral exchange rates result in a biased estimator of the regression coefficients (Kunkler, 2021). Two of the most popular multicurrency numéraires are the International Monetary Fund's (IMF) Special Drawing Right (SDR) that was created in 1969 and the trade-weighted basket of numéraire currencies used for the US dollar index that was created in 1973. Another multicurrency numéraire that results in numéraire-invariant multilateral exchange rates is an equally-weighted basket of numéraire currencies

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(Hovanov et al., 2004; Kunkler & MacDonald, 2015).

A multicurrency numéraire is a weighted basket of numéraire currencies. The numéraire currencies themselves can be priced in terms of the multicurrency numéraire, even though the numéraire currencies are members of the multicurrency numéraire. For example, the US dollar can be priced in terms of the Special Drawing Right (SDR), even though the US dollar is a numéraire currency of the SDR. A no-arbitrage condition enforces the log returns of the system of multilateral exchange rates associated with the numéraire currencies to be a singular system, where the covariance matrix is singular and its ordinary inverse does not exist. Singular systems pose a methodological challenge when it comes to multivariate regression models. For example, when the disturbance covariance matrix from the multivariate regression model is singular, the ordinary matrix inverse does not exist. As a consequence, the generalized least squares (GLS) estimator of the regression coefficients also does not exist.

Much of the work in this area has been stimulated by the work on allocation models (see Haupt & Oberhofer, 2006, and the references within). There are two main solutions, namely, the deletion of one of the equations (Barten, 1969), or modifying the generalized least squares estimator of the regression coefficients (Theil, 1971). The former solution leads to the question of whether the deleted equation impacts the estimated parameters (see Bewley, 1986; Fiebig, 2001; Haupt & Oberhofer, 2006). The latter solution involves modifying the generalized least squares (GLS) estimator of the regression coefficients by replacing the ordinary inverse of the disturbance covariance matrix with the Moore-Penrose generalized inverse of the disturbance covariance (see Kreijger & Neudecker, 1977; Theil, 1971).

The solution that modifies the generalized least squares estimator of the regression coefficient (Theil, 1971) requires a singular disturbance covariance matrix. However, using any set of explanatory variables does not necessarily result in a singular disturbance covariance matrix. There are necessary and sufficient conditions for the existence of the generalized-inverse estimator of the regression coefficients (Dhrymes & Schwarz, 1987a). As a consequence, implicit restrictions are required on the explanatory variables and the regression coefficients to enforce the disturbance covariance matrix to be singular. If the implicit restrictions are satisfied, the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

This paper contributes to the literature by highlighting the methodological challenge of estimating regression coefficients when the numéraire currencies of a multicurrency numéraire are the dependent variables in a multivariate regression model. A solution is proposed based on Theil (1971) that modifies the generalized least squares (GLS) estimator of the regression coefficients by replacing the ordinary inverse with the generalized inverse. However, certain conditions need to be satisfied for the disturbance covariance matrix from the multivariate regression model to be singular. As a result, the solution imposes implicit restrictions on the explanatory variables and the regression coefficients to guarantee a singular disturbance covariance matrix. The multivariate regression framework can be classified as a model-derived singularity (Haupt & Oberhofer, 2002). The solution provides a consistent multivariate regression model to explain the observed heterogeneity in the relative currency market.

Seven developed market currencies are considered, namely, the US dollar, the Japanese yen, the Eurozone euro, the Swiss franc, the British pound, the Canadian dollar, and the Australian dollar. The proposed multivariate regression framework does not pre-specify which explanatory variables should be used in a multivariate regression model: similar to the Arbitrage Pricing Theory (see Ross, 1976). In addition, choosing explanatory variables for a multivariate regression model is a non-trivial task. For example, it is well-known that there is an exchange-rate disconnect puzzle between exchange rates and fundamentals (Engel & West, 2005; Meese & Rogoff, 1983a, 1983b). Thus, choosing the best set of explanatory variables is beyond the scope of this paper and is left for future research.

A focal point of this paper is to illustrate the implicit restrictions and to provide an application of the proposed multivariate regression framework. Currencies have been shown to move heterogeneously with global equity markets (Campbell et al., 2010). As a consequence, a group of local equity market indexes is used as explanatory variables, where there is one local equity market index associated with each currency. The results section confirms the previously reported heterogeneity in the currency market by showing that all currencies cannot move with, or against, the movements in a global equity market index. Furthermore, all currencies move in a common (panel-like) fashion with, or against, to the movements in a group of relative (idiosyncratic) equity market indexes.

The rest of this paper is organized as follows. Section 2 provides the material and methods. Section 3 describes the data sample and presents the results, and Section 4 concludes.

2. Material and methods

2.1. Bilateral exchange rates

A bilateral exchange rate is the price of one currency in terms of another currency: a single-currency numéraire. In log terms,[1] for a system of N_C currencies, let $p_{B,t}^{i/j}$ represent the ith/jth bilateral exchange rate at time t for the ith currency in terms of the jth currency, where $i,j = 1,\dots,N_C$; and $t = 0,\dots,T$. There is a no-arbitrage condition between the bilateral exchange rates for any three currencies given by:

$$p_{B,t}^{i/j} = p_{B,t}^{i/k} - p_{B,t}^{j/k} \quad (1)$$

[1] Bilateral exchange rates are modelled in log terms so the model does not suffer from the well-known Siegel Paradox (Siegel, 1972).

where $i,j,k = 1,\ldots,N_C$; $t = 0,\ldots,T$; $p_{B,t}^{i/j}$ is the ith/jth bilateral exchange rate; $p_{B,t}^{i/k}$ is the ith/kth bilateral exchange rate; and $p_{B,t}^{j/k}$ is the jth/kth bilateral exchange rate (see Chacholiades, 1971). The no-arbitrage condition in (1) can be rearranged to give:

$$p_{B,t}^{i/j} + p_{B,t}^{k/i} + p_{B,t}^{j/k} = 0 \tag{2}$$

where $i,j,k = 1,\ldots,N_C$; $t = 0,\ldots,T$; $p_{B,t}^{i/j}$ is the ith/jth bilateral exchange rate; $p_{B,t}^{k/i}$ is the kth/ith bilateral exchange rate with $p_{B,t}^{k/i} = -p_{B,t}^{i/k}$; and $p_{B,t}^{j/k}$ is the jth/kth bilateral exchange rate. The no-arbitrage condition in (2) also applies to the log returns of the bilateral exchange rates for any three currencies given by:

$$\Delta p_{B,t}^{i/j} + \Delta p_{B,t}^{k/i} + \Delta p_{B,t}^{j/k} = 0 \tag{3}$$

where $i,j,k = 1,\ldots,N_C$; $t = 1,\ldots,T$; $\Delta p_{B,t}^{i/j} = p_{B,t}^{i/j} - p_{B,t-1}^{i/j}$ is the log return of the ith/jth bilateral exchange rate; $\Delta p_{B,t}^{k/i} = p_{B,t}^{k/i} - p_{B,t-1}^{k/i}$ is the log return of the kth/ith bilateral exchange rate; $\Delta p_{B,t}^{j/k} = p_{B,t}^{j/k} - p_{B,t-1}^{j/k}$ is the log return of the jth/kth exchange rate; and Δ is the first difference operator.

2.2. Multilateral exchange rates

A multilateral exchange rate is the price of one currency in terms of a weighted basket of numéraire currencies: a multicurrency numéraire. In log terms, for a system of N_C currencies, each multilateral exchange rate can be written as:

$$p_{M,t}^{i} = \sum\nolimits_{j=1}^{N_C} w^j p_{B,t}^{i/j} \tag{4}$$

where $i = 1,\ldots,N_C$; $t = 0,\ldots,T$; $p_{M,t}^{i}$ is the ith multilateral exchange rate for the ith currency; w^j is the numéraire weight associated with the jth numéraire currency; and $p_{B,t}^{i/j}$ is the ith/jth bilateral exchange rate, with $p_{B,t}^{i/i} = 0$. It should be noted that the ith currency is also a numéraire currency of the multicurrency numéraire. For example, the US dollar is a numéraire currency of the Special Drawing Right (SDR) from the International Monetary Fund (IMF). The movements in the US dollar can be priced in terms of the SDR by:

$$\Delta p_{M,t}^{USD} = \sum_{j=1}^{5} w^j \Delta p_{B,t}^{USD/j} = 0.4338\Delta p_{B,t}^{USD/USD} + 0.2931\Delta p_{B,t}^{USD/EUR} + 0.1228\Delta p_{B,t}^{USD/CNY} + 0.0759\Delta p_{B,t}^{USD/JPY} + 0.0744\Delta p_{B,t}^{USD/GBP} \tag{5}$$

where $t = 1,\ldots,T$; $\Delta p_{M,t}^{USD}$ is the log return of the US dollar multilateral exchange rate in terms of the SDR; and $\Delta p_{B,t}^{USD/j}$ is the log return of the USD/jth bilateral exchange rate with $j = 1,\ldots,5$ and $\Delta p_{B,t}^{USD/USD} = 0$. The numéraire weights of the SDR are associated with the review in 2022, where the weights for the five numéraire currencies are: 0.4338 for the US dollar (USD), 0.2931 for the Eurozone euro (EUR), 0.1228 for the Chinese renminbi (CNY), 0.0759 for the Japanese yen (JPY), and 0.0744 for the British pound (GBP). Note that the numéraire weights of the SDR sum to one.

It is assumed throughout the rest of this paper that multilateral exchange rates are priced relative to an equally-weighted basket of N_C numéraire currencies so that $w^j = 1/N_C$ for $j = 1,\ldots,N_C$. In this case, each multilateral exchange rate in (4) can be rewritten as:

$$p_{M,t}^{i} = \frac{1}{N_C}\sum\nolimits_{j=1}^{N_C} p_{B,t}^{i/j} \tag{6}$$

where $i = 1,\ldots,N_C$; $t = 0,\ldots,T$; $p_{M,t}^{i}$ is the ith multilateral exchange rate for the ith currency; and $p_{B,t}^{i/j}$ is the ith/jth bilateral exchange rate with $j = 1,\ldots,N_C$.

2.3. Decomposing bilateral exchange rates

Bilateral exchange rates can be decomposed into the difference between two of the multilateral exchange rates from (6) by:

$$p_{B,t}^{i/j} = p_{M,t}^{i} - p_{M,t}^{j} \tag{7}$$

where $i,j = 1,\ldots,N_C$, $t = 0,\ldots,T$; $p_{B,t}^{i/j}$ is the ith/jth bilateral exchange rate; $p_{M,t}^{i}$ is the ith multilateral exchange rate; and $p_{M,t}^{j}$ is the jth multilateral exchange rate (Kunkler & MacDonald, 2015). The decomposition in (7) also applies to the log returns of the bilateral exchange rates by:

$$\Delta p_{B,t}^{i/j} = \Delta p_{M,t}^{i} - \Delta p_{M,t}^{j} \tag{8}$$

where $i,j = 1,\ldots,N_C$; $t = 1,\ldots,T$; $\Delta p_{B,t}^{i/j} = p_{B,t}^{i/j} - p_{B,t-1}^{i/j}$ is the log return of the ith/jth bilateral exchange rate; $\Delta p_{M,t}^{i} = p_{M,t}^{i} - p_{M,t-1}^{i}$ is the log return of the ith multilateral exchange rate; and $\Delta p_{M,t}^{j} = p_{M,t}^{j} - p_{M,t-1}^{j}$ is the log return of the jth multilateral exchange rate.

2.4. A multivariate regression model

In general, a multivariate regression model for the log returns of the N_C multilateral exchange rates with N_P explanatory variables can be written as:

$$\Delta \boldsymbol{p}_M^i = \sum\nolimits_{p=1}^{N_P} \boldsymbol{x}_M^{p,i} \beta_M^{p,i} + \boldsymbol{\varepsilon}_M^i \tag{9}$$

where $i = 1, \ldots, N_C$; $\Delta \boldsymbol{p}_M^i$ is a $T \times 1$ vector of log returns of the ith multilateral exchange rate; $\boldsymbol{x}_M^{p,i}$ is a $T \times 1$ vector for the pth explanatory variable for the ith multilateral exchange rate; $\beta_M^{p,i}$ is the ith regression coefficient for the pth explanatory variable for the ith multilateral exchange rate; and $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term.

2.5. Singular system of equations

This section shows that the log returns for the system of N_C multilateral exchange rates for the N_C numéraire currencies is a singular system. In matrix notation, there is a no-arbitrage condition for the N_C multilateral exchange rates in (6) such that:

$$\sum\nolimits_{i=1}^{N_C} \boldsymbol{p}_M^i = \boldsymbol{0} \tag{10}$$

where $\boldsymbol{p}_M^i$ is a $(T+1) \times 1$ vector of the ith multilateral exchange rate; and $\boldsymbol{0}$ is a $(T+1) \times 1$ vector of zeros (Kunkler & MacDonald, 2015). The no-arbitrage condition in (10) also applies to the log returns of the multilateral exchange rates by:

$$\sum\nolimits_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i = \boldsymbol{0} \tag{11}$$

where $\Delta \boldsymbol{p}_M^i$ is a $T \times 1$ vector of the log returns of the ith multilateral exchange rate; $\Delta p_{M,t}^i = p_{M,t}^i - p_{M,t-1}^i$ is the log return of the ith multilateral exchange rate at time t with $i = 1, \ldots, N_C$; and $\boldsymbol{0}$ is a $T \times 1$ vector of zeros.

The no-arbitrage condition for the log returns of the system of N_C multilateral exchange rates in (11) creates a linear dependency, which can be seen by rearranging (11) to give:

$$\Delta \boldsymbol{p}_M^i = -\sum\nolimits_{j=1}^{N_C} I(i \neq j) \Delta \boldsymbol{p}_M^j \tag{12}$$

where $i = 1, \ldots, N_C$; $\Delta \boldsymbol{p}_M^i$ is a $T \times 1$ vector of the log returns of the ith multilateral exchange rate; $\Delta \boldsymbol{p}_M^j$ is a $T \times 1$ vector of the log returns of the jth multilateral exchange rate; and $I(i \neq j)$ is an indicator variable that is one when $i \neq j$ and zero otherwise.

The linear dependency between the log returns of the system of N_C multilateral exchange rates results in a singular covariance matrix of the log returns of the multilateral exchange rates, where the ordinary matrix inverse does not exist. Let the covariance matrix be represented by:

$$\boldsymbol{\Sigma}_{\Delta \boldsymbol{p}} = \operatorname{cov}\left(\Delta \boldsymbol{p}_M^1, \ldots, \Delta \boldsymbol{p}_M^{N_C}\right) \tag{13}$$

where $\boldsymbol{\Sigma}_{\Delta \boldsymbol{p}}$ is an $N_C \times N_C$ covariance matrix for the log returns of the system of N_C multilateral exchange rates; and $\sigma_{\Delta \boldsymbol{p}}^{i,j} = \operatorname{cov}\left(\Delta \boldsymbol{p}_M^i, \Delta \boldsymbol{p}_M^j\right)$ is the covariance between $\Delta \boldsymbol{p}_M^i$ and $\Delta \boldsymbol{p}_M^j$, and represents the ith row and jth column of $\boldsymbol{\Sigma}_{\Delta \boldsymbol{p}}$, with $i,j = 1, \ldots, N_C$.

In summary, the log returns of the system of N_C multilateral exchange rates for the N_C numéraire currencies is a singular system. Thus, the covariance matrix of the log returns for the system of N_C multilateral exchange rates is singular and the ordinary inverse does not exist. Singular systems pose a challenge when it comes to multivariate regression analysis. For example, when the disturbance covariance matrix is also singular, the ordinary matrix inverse does not exist and consequently the generalized least squares (GLS) estimator of the regression coefficients will not exist. Theil (1971) suggested that the generalized inverse of the disturbance covariance matrix replace the ordinary inverse. Under these circumstances, there are necessary and sufficient conditions on the existence of the generalized-inverse estimator of the regression coefficients (Dhrymes & Schwarz, 1987a). In addition, if the adding up conditions are imposed the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

2.6. Implicit restrictions

2.6.1. Introduction

Using any set of explanatory variables does not necessarily result in a singular disturbance covariance matrix. Consequently, implicit restrictions are imposed on the explanatory variables and the regression coefficients, so the disturbance covariance matrix from the multivariate regression is singular. If the implicit restrictions are satisfied, the generalized-inverse estimator of the regression coefficients will always exist (Dhrymes & Schwarz, 1987b).

The covariance matrix of the system of N_C disturbance terms in (9) is given by:

$$\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \operatorname{cov}\left(\boldsymbol{\varepsilon}_M^1, \ldots, \boldsymbol{\varepsilon}_M^{N_C}\right) \tag{14}$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$ is an $N_C \times N_C$ disturbance covariance matrix; and $\sigma_{\boldsymbol{\varepsilon}}^{i,j} = \operatorname{cov}\left(\boldsymbol{\varepsilon}_M^i, \boldsymbol{\varepsilon}_M^j\right)$ is the covariance between $\boldsymbol{\varepsilon}_M^i$ and $\boldsymbol{\varepsilon}_M^j$, and represents the ith row and jth column of $\boldsymbol{\Sigma}_{\varepsilon}$, with $i,j = 1,\ldots,N_C$. One way for the disturbance covariance matrix to be singular is to impose implicit restrictions on the explanatory variables and the regression coefficients so that:

$$\sum\nolimits_{i=1}^{N_C}\boldsymbol{\varepsilon}_M^i = \mathbf{0} \tag{15}$$

where $\boldsymbol{\varepsilon}_M^i$ is a $T\times1$ vector of the ith disturbance term with $i = 1,\ldots,N_C$; and $\mathbf{0}$ is a $T\times1$ vector of zeros.

Thus, implicit restrictions can be imposed on the explanatory variables and the regression coefficients so that the disturbance covariance matrix will be singular. Subsequently, the generalized least squares (GLS) estimator of the regression coefficients can be modified by replacing the ordinary inverse with the generalized-inverse (Theil, 1971). If the implicit restrictions are satisfied, the disturbance covariance matrix will be singular. In addition, the generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients will always exist.

Different implicit restrictions are required for different explanatory variables. It is assumed that explanatory variables belong to one of three main types, namely, absolute explanatory variables, a group of relative explanatory variables, or a group of hybrid explanatory variables. Each of the three main types of explanatory variables, together with the implicit restrictions, are discussed in the subsections below.

2.6.2. Absolute explanatory variables

An absolute explanatory variable is a common explanatory variable for all currencies. Each currency has a separate regression coefficient associated with the absolute explanatory variable. For example, the movements in a global equity market index is an absolute explanatory variable, as there is only one global equity market index and all currencies have their own exposure to the movements in the global equity market index. It is assumed that there are N_A absolute explanatory variables in a multivariate regression model.

The implicit regression-coefficient restriction for an absolute explanatory variable is that the sum of the regression coefficients across the system of N_C currencies is equal to zero:

$$\sum\nolimits_{i=1}^{N_C}\beta_A^{a,i} = 0 \tag{16}$$

where $a = 1,\ldots,N_A$; and $\beta_A^{a,i}$ is the ith regression coefficient associated for the ath absolute explanatory variable with $i = 1,\ldots,N_C$.

In matrix form, a multivariate regression model for an absolute explanatory variable ($N_A = 1$) and the associated regression coefficients can be written by stacking the system of N_C equations to give:

$$\Delta\boldsymbol{p}_M = \begin{bmatrix}\Delta\boldsymbol{p}_M^1\\ \vdots\\ \vdots\\ \Delta\boldsymbol{p}_M^{N_C}\end{bmatrix} = \begin{bmatrix}\boldsymbol{x}_A^a & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \mathbf{0}\\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{x}_A^a\end{bmatrix}\begin{bmatrix}\beta_A^{a,1}\\ \vdots\\ \beta_A^{a,N_C}\end{bmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}_M^1\\ \vdots\\ \vdots\\ \boldsymbol{\varepsilon}_M^{N_C}\end{bmatrix} = \boldsymbol{X}_A^a\boldsymbol{\beta}_A^a + \boldsymbol{\varepsilon}_M \tag{17}$$

where $a = 1$; $\Delta\boldsymbol{p}_M = (\Delta\boldsymbol{p}_M^1;\cdots;\Delta\boldsymbol{p}_M^{N_C})$ is a $(N_C*T)\times1$ stacked vector of the log returns of the system of N_C multilateral exchange rates; $\boldsymbol{x}_A^a$ is a $T\times1$ vector of the ath absolute explanatory variable; $\beta_A^{a,i}$ is the ith regression coefficient associated for the ath absolute explanatory variable with $i = 1,\ldots,N_C$; $\boldsymbol{X}_A^a = \mathrm{diag}(\boldsymbol{x}_A^a,\ldots,\boldsymbol{x}_A^a)$ is a $(N_C*T)\times N_C$ diagonal matrix for ath absolute explanatory variable; $\boldsymbol{\beta}_A^a = (\beta_A^{a,1};\cdots;\beta_A^{a,N_C})$ is a $N_C\times1$ vector the regression coefficients associated for the ath absolute explanatory variable; and $\boldsymbol{\varepsilon}_M = (\boldsymbol{\varepsilon}_M^1;\cdots;\boldsymbol{\varepsilon}_M^{N_C})$ is a $(N_C*T)\times1$ stacked vector of disturbance terms.

If the implicit restriction for the absolute explanatory variable in (16) is satisfied, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C}\boldsymbol{\varepsilon}_M^i = \sum\nolimits_{i=1}^{N_C}\Delta\boldsymbol{p}_M^i - \sum\nolimits_{i=1}^{N_C}\boldsymbol{x}_A^a\beta_A^{a,i} = \sum\nolimits_{i=1}^{N_C}\Delta\boldsymbol{p}_M^i - \boldsymbol{x}_A^a\sum\nolimits_{i=1}^{N_C}\beta_A^{a,i} = \boldsymbol{0} \tag{18}$$

where $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term in (17); $\sum_{i=1}^{N_C}\Delta\boldsymbol{p}_M^i = \boldsymbol{0}$ from (11); and $\sum_{i=1}^{N_C}\beta_A^{a,i} = 0$ from (16). In this situation, the disturbance covariance matrix $\boldsymbol{\Sigma}_{\varepsilon}$ is singular.

2.6.3. Relative explanatory variables

A relative explanatory variable is an idiosyncratic explanatory variable for a specific currency. The relative explanatory variable belongs to a group of N_C relative explanatory variables, where there is one for each currency. For example, the group of lagged log returns of the N_C multilateral exchange rates is a group of relative explanatory variables. It is assumed that there are N_R groups of relative explanatory variables in a multivariate regression model, where each group contains N_C relative explanatory variables: one for each currency.

The implicit regression-coefficient restriction for each group of relative explanatory variables is that all currencies share a common (pooled) regression coefficient, represented by β_R^r for the rth group of relative explanatory variables, where $r = 1,\ldots,N_R$. In addition, the implicit explanatory-variable restriction for each group of relative explanatory variables is that the cross-sectional sum, at each time t, is equal to zero:

$$\sum_{i=1}^{N_C} x_{R,t}^{r,i} = 0 \tag{19}$$

where $r = 1, \ldots, N_R$, $t = 1, \ldots, T$; and $x_{R,t}^{r,i}$ is the ith relative explanatory variable in the rth group of relative explanatory variables.

In matrix form, a multivariate regression model for a group of relative explanatory variables ($N_R = 1$) and the associated common regression coefficient can be written by stacking the system of N_C equations to give:

$$\Delta \boldsymbol{p}_M = \begin{bmatrix} \Delta \boldsymbol{p}_M^1 \\ \vdots \\ \Delta \boldsymbol{p}_M^{N_C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_R^{r,1} \\ \vdots \\ \boldsymbol{x}_R^{r,N_C} \end{bmatrix} \beta_R^r + \begin{bmatrix} \boldsymbol{\varepsilon}_M^1 \\ \vdots \\ \boldsymbol{\varepsilon}_M^{N_C} \end{bmatrix} = \boldsymbol{X}_R^r \beta_R^r + \boldsymbol{\varepsilon}_M \tag{20}$$

where $r = 1$; $\Delta \boldsymbol{p}_M = (\Delta \boldsymbol{p}_M^1; \cdots; \Delta \boldsymbol{p}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of the log returns of the system of N_C multilateral exchange rates; $\boldsymbol{x}_R^{r,i}$ is a $T \times 1$ vector of the ith relative explanatory variable for the rth group of relative explanatory variables with $i = 1, \ldots, N_C$; β_R^r is a common regression coefficient associated for the rth group of relative explanatory variables; $\boldsymbol{X}_R^r = (\boldsymbol{x}_R^{r,1}; \cdots; \boldsymbol{x}_R^{r,N_C})$ is a $(N_C * T) \times 1$ stacked vector of the rth group of relative explanatory variables; and $\boldsymbol{\varepsilon}_M = (\boldsymbol{\varepsilon}_M^1; \cdots; \boldsymbol{\varepsilon}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of disturbance terms.

If the implicit restrictions are satisfied for a group of relative explanatory variables, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \boldsymbol{\varepsilon}_M^i = \sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \sum_{i=1}^{N_C} \boldsymbol{x}_R^{r,i} \beta_R^r = \sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \beta_R^r \sum_{i=1}^{N_C} \boldsymbol{x}_R^{r,i} = \boldsymbol{0} \tag{21}$$

where $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term in (20); $\sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i = \boldsymbol{0}$ from (11); and $\sum_{i=1}^{N_C} \boldsymbol{x}_R^{r,i} = \boldsymbol{0}$ from (19). In this situation, the disturbance covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$ is singular.

2.6.4. Hybrid explanatory variables

Hybrid explanatory variables are similar to relative explanatory variables. For example, a hybrid explanatory variable is a currency-specific explanatory variable and belongs to a group of hybrid explanatory variables. For example, the movements in local equity market indexes are a group of hybrid explanatory variables, where there is one local equity market index associated with each currency. It is assumed that there are N_H groups of hybrid explanatory variables in the multivariate regression model, where each group contains N_H hybrid explanatory variables: one for each currency.

The difference between a group of hybrid explanatory variables and a group of relative explanatory variables is that the cross-sectional sum for a group of hybrid explanatory variables, at each time t, does not equal to zero:

$$\sum_{i=1}^{N_C} x_{H,t}^{h,i} \neq 0 \tag{22}$$

where $h = 1, \ldots, N_H$; $t = 1, \ldots, T$; and $x_{H,t}^{h,i}$ is the ith hybrid explanatory variable in the hth group of hybrid explanatory variables with $i = 1, \ldots, N_C$.

The implicit explanatory-variable restriction decomposes each group of N_C hybrid explanatory variables into an absolute explanatory variable and a group of N_C relative explanatory variables by:

$$x_{H,t}^{h,i} = x_{A,t}^{h} + x_{R,t}^{h,i} \tag{23}$$

where $h = 1, \ldots, N_H$; $i = 1, \ldots, N_C$; $t = 1, \ldots, T$; $x_{H,t}^{h,i}$ is the ith hybrid explanatory variable in the hth group of hybrid explanatory variables; $x_{A,t}^{h}$ is the absolute explanatory variable for the hth group; and $x_{R,t}^{h,i}$ is the ith relative explanatory variable for the hth group.

The absolute explanatory variable in (23) can be created by calculating the cross-sectional average of the group of hybrid explanatory variables to give:

$$x_{A,t}^{h} = \frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{h,i} \tag{24}$$

where $h = 1, \ldots, N_H$; $t = 1, \ldots, T$; $x_{A,t}^{h}$ is the absolute explanatory variable for the hth group; and $x_{H,t}^{h,i}$ is the ith hybrid explanatory variable for the hth group. Subsequently, the relative explanatory variables in (23) can be calculated by subtracting the absolute explanatory variable for the hth group in (24) from the ith hybrid explanatory variable for the hth group to give:

$$x_{R,t}^{h,i} = x_{H,t}^{h,i} - x_{A,t}^{h} \tag{25}$$

where $h = 1, \ldots, N_H$; $t = 1, \ldots, T$; $x_{R,t}^{h,i}$ is the ith relative explanatory variable for the hth group; $x_{H,t}^{h,i}$ is the ith hybrid explanatory variable for the hth group; and $x_{A,t}^{h}$ is the absolute explanatory variable for the hth group in (24).

The cross-sectional sum of the group of relative explanatory variables in (25) is:

$$\sum_{i=1}^{N_C} x_{R,t}^{h,i} = \sum_{i=1}^{N_C} \left(x_{H,t}^{h,i} - x_{A,t}^{h} \right) = \sum_{i=1}^{N_C} x_{H,t}^{h,i} - N_C \left(\frac{1}{N_C} \sum_{i=1}^{N_C} x_{H,t}^{h,i} \right) = 0 \tag{26}$$

where $h = 1,\ldots,N_H$; $t = 1,\ldots,T$; $x_{R,t}^{h,i} = x_{H,t}^{h,i} - x_{A,t}^{h}$ is the ith relative explanatory variable for the hth group in (25); $x_{H,t}^{h,i}$ is the ith hybrid explanatory variable for the hth group; and $x_{A,t}^{h} = \frac{1}{N_C}\sum_{i=1}^{N_C} x_{H,t}^{h,i}$ is the absolute explanatory variable for the hth group in (24). Thus, the group of relative explanatory variables for the hth group in (25) satisfies the implicit explanatory-variable restriction in (19), where the cross-sectional sum of the group of relative explanatory variables is equal to zero.

In matrix form, a multivariate regression model for the decomposition of a group of N_C hybrid explanatory variables ($N_H = 1$) into an absolute explanatory variable and a group of N_C relative explanatory variables can be written by stacking the system of N_C equations to give:

$$\Delta \boldsymbol{p}_M = \begin{bmatrix} \Delta \boldsymbol{p}_M^1 \\ \vdots \\ \vdots \\ \Delta \boldsymbol{p}_M^{N_C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_A^h & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{x}_A^h \end{bmatrix} \begin{bmatrix} \beta_A^{h,1} \\ \vdots \\ \beta_A^{h,N_C} \end{bmatrix} + \begin{bmatrix} \boldsymbol{x}_R^{h,1} \\ \vdots \\ \boldsymbol{x}_R^{h,N_C} \end{bmatrix} \beta_R^h + \begin{bmatrix} \boldsymbol{\varepsilon}_M^1 \\ \vdots \\ \vdots \\ \boldsymbol{\varepsilon}_M^{N_C} \end{bmatrix} = \boldsymbol{X}_A^h \boldsymbol{\beta}_A^h + \boldsymbol{X}_R^h \beta_R^h + \boldsymbol{\varepsilon}_M \tag{27}$$

where $h = 1$; $\Delta \boldsymbol{p}_M = (\Delta \boldsymbol{p}_M^1; \cdots; \Delta \boldsymbol{p}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of the log returns of the system of N_C multilateral exchange rates; $\boldsymbol{x}_A^h$ is a $T \times 1$ vector of the absolute explanatory variable for the hth group; $\boldsymbol{X}_A^h = \mathrm{diag}(\boldsymbol{x}_A^h, \ldots, \boldsymbol{x}_A^h)$ is a $(N_C * T) \times N_C$ diagonal matrix for the absolute explanatory variable for the hth group; $\boldsymbol{\beta}_A^h = (\beta_A^{h,1}; \cdots; \beta_A^{h,N_C})$ is a $N_C \times 1$ vector the regression coefficients for the hth group; $\beta_A^{h,i}$ is the ith regression coefficient for the hth group with $i = 1,\ldots,N_C$; $\boldsymbol{x}_R^{h,i}$ is a $T \times 1$ vector of the ith relative explanatory variable for the hth group; β_R^h is the regression coefficient associated for the hth group; $\boldsymbol{X}_R^h = (\boldsymbol{x}_R^{h,1}; \cdots; \boldsymbol{x}_R^{h,N_C})$ is a $(N_C * T) \times 1$ stacked vector of the hth group of relative explanatory variables; and $\boldsymbol{\varepsilon}_M = (\boldsymbol{\varepsilon}_M^1; \cdots; \boldsymbol{\varepsilon}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of disturbance terms.

If the implicit restrictions are satisfied for the group of hybrid explanatory variables, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \boldsymbol{\varepsilon}_M^i = \sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \sum_{i=1}^{N_C} \left(\boldsymbol{x}_A^h \beta_A^{h,i} + \boldsymbol{x}_R^{h,i} \beta_R^h \right) = \sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \boldsymbol{x}_A^h \sum_{i=1}^{N_C} \beta_A^{h,i} + \beta_R^h \sum_{i=1}^{N_C} \boldsymbol{x}_R^{h,i} = \boldsymbol{0} \tag{28}$$

where $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term in (27); $\sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i = \boldsymbol{0}$ from (11); $\sum_{i=1}^{N_C} \beta_A^{h,i} = 0$ from (16); and $\sum_{i=1}^{N_C} \boldsymbol{x}_R^{h,i} = \boldsymbol{0}$ from (19). In this situation, the disturbance covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$ is singular.

2.6.5. Summary

In general, explanatory variables can be classified into one of three main types, namely, absolute explanatory variables, a group of relative explanatory variables, or a group of hybrid explanatory variables. However, a group of hybrid explanatory variables can be decomposed into an absolute explanatory variable and a group of relative explanatory variables. Thus, each explanatory variable in a multivariate regression model is either an absolute explanatory variable or a relative explanatory variable. As a result, there are only two core types of explanatory variables, namely, absolute explanatory variables and relative explanatory variables.

2.7. A multivariate regression model with implicit restrictions

The form of a multivariate regression model for the log returns of each multilateral exchange rate with N_A absolute explanatory variables (including one for the intercept) and N_R groups of relative explanatory variables can be written as:

$$\Delta \boldsymbol{p}_M^i = \sum_{a=1}^{N_A} \boldsymbol{x}_A^a \beta_A^{a,i} + \sum_{r=1}^{N_R} \boldsymbol{x}_R^{r,i} \beta_R^r + \boldsymbol{\varepsilon}_M^i \tag{29}$$

where $i = 1,\ldots,N_C$; $\Delta \boldsymbol{p}_M^i$ is a $T \times 1$ vector of log returns of the ith multilateral exchange rate; $\boldsymbol{x}_A^a$ is a $T \times 1$ vector for the ath absolute explanatory variable; $\beta_A^{a,i}$ is the ith regression coefficient for the ath absolute explanatory variable; $\boldsymbol{x}_R^{r,i}$ is a $T \times 1$ vector for the ith explanatory variable for the rth group of relative explanatory variables; β_R^r is the common regression coefficient for the rth group of relative explanatory variables; and $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term. It should be noted that the multivariate regression model in (29) also includes the hybrid explanatory variables, subsequent to the decomposition in (23).

In matrix form, the multivariate regression model in (29) can be written by stacking the system of N_C equations to give:

$$\Delta \boldsymbol{p}_M = \begin{bmatrix} \Delta \boldsymbol{p}_M^1 \\ \vdots \\ \Delta \boldsymbol{p}_M^{N_C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_A^1 & \cdots & \boldsymbol{X}_A^{N_A} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_A^1 \\ \vdots \\ \boldsymbol{\beta}_A^{N_A} \end{bmatrix} + \begin{bmatrix} \boldsymbol{X}_R^1 & \cdots & \boldsymbol{X}_R^{N_R} \end{bmatrix} \begin{bmatrix} \beta_R^1 \\ \vdots \\ \beta_R^{N_R} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_M^1 \\ \vdots \\ \boldsymbol{\varepsilon}_M^{N_C} \end{bmatrix} = \boldsymbol{X}_A \boldsymbol{\beta}_A + \boldsymbol{X}_R \boldsymbol{\beta}_R + \boldsymbol{\varepsilon}_M \tag{30}$$

where $\Delta \boldsymbol{p}_M = (\Delta \boldsymbol{p}_M^1; \cdots; \Delta \boldsymbol{p}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of the log returns of the system of N_C multilateral exchange rates; $\boldsymbol{X}_A^a =$

$\mathrm{diag}(\boldsymbol{x}_A^a, \ldots, \boldsymbol{x}_A^a)$ is a $(N_C * T) \times N_C$ diagonal matrix for the ath absolute explanatory variables in (17) with $a = 1, \ldots, N_A$; $\boldsymbol{\beta}_A^a$ is a $N_C \times 1$ vector regression coefficients for the ath absolute explanatory variables in (17); $\boldsymbol{X}_R^r$ is a $(N_C * T) \times 1$ stacked vector for the rth group of relative explanatory variables in (20) with $r = 1, \ldots, N_R$; β_R^r is the common (panel-like) regression coefficient for the rth group of relative explanatory variables in (20); and $\boldsymbol{\varepsilon}_M = (\boldsymbol{\varepsilon}_M^1; \cdots; \boldsymbol{\varepsilon}_M^{N_C})$ is a $(N_C * T) \times 1$ stacked vector of disturbance terms.

If the implicit restrictions are satisfied, the cross-sectional sum of the disturbance terms is given by:

$$\sum_{i=1}^{N_C} \boldsymbol{\varepsilon}_M^i = \sum\nolimits_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \sum\nolimits_{i=1}^{N_C} \left(\sum\nolimits_{a=1}^{N_A} \boldsymbol{x}_A^a \beta_A^{a,i} + \sum\nolimits_{r=1}^{N_R} \boldsymbol{x}_R^{r,i} \beta_R^r \right) = \sum\nolimits_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i - \sum\nolimits_{a=1}^{N_A} \boldsymbol{x}_A^a \sum\nolimits_{i=1}^{N_C} \beta_A^{a,i} + \sum\nolimits_{r=1}^{N_R} \beta_R^r \sum\nolimits_{i=1}^{N_C} \boldsymbol{x}_R^{r,i} = \boldsymbol{0} \tag{31}$$

where $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term in (30); $\sum_{i=1}^{N_C} \Delta \boldsymbol{p}_M^i = \boldsymbol{0}$ from (11); $\sum_{i=1}^{N_C} \beta_A^{a,i} = 0$ from (16); and $\sum_{i=1}^{N_C} \boldsymbol{x}_R^{r,i} = \boldsymbol{0}$ from (19). In this situation, the disturbance covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}$ is singular.

The full covariance matrix of the disturbance terms is given by:

$$\boldsymbol{\Omega}_\varepsilon = \boldsymbol{\Sigma}_\varepsilon \bigotimes \mathbf{I}_T \tag{32}$$

where $\boldsymbol{\Omega}_\varepsilon$ is a $(N_C * T) \times (N_C * T)$ full covariance matrix; $\boldsymbol{\Sigma}_\varepsilon$ is an $N_C \times N_C$ disturbance covariance matrix in (14); and $\bigotimes$ is the Kronecker product. The generalized least squares (GLS) estimator of the regression coefficients of the multivariate regression in (30) is given by:

$$E(\boldsymbol{\beta}) = \left(\boldsymbol{X}' \boldsymbol{\Omega}_\varepsilon^{-1} \boldsymbol{X}\right)^{-1} \left(\boldsymbol{X}' \boldsymbol{\Omega}_\varepsilon^{-1} \boldsymbol{y}\right). \tag{33}$$

where $\boldsymbol{\Omega}_\varepsilon$ is a $(N_C * T) \times (N_C * T)$ full covariance matrix of the disturbance terms in (32); $\boldsymbol{X} = [\boldsymbol{X}_A \quad \boldsymbol{X}_{\boldsymbol{R}}]$ is a $(N_C * T) \times (N_A * N_C + N_R)$ matrix of the combined explanatory variables, with N_A absolute explanatory variables and N_R groups of relative explanatory variables; and $\boldsymbol{\beta} = (\boldsymbol{\beta}_A; \boldsymbol{\beta}_R)$ is a $(N_A * N_C + N_R) \times 1$ vector of the combined regression coefficients.

However, the full disturbance covariance matrix in (32) is singular from (31) so the ordinary inverse of the disturbance covariance matrix $\boldsymbol{\Omega}_\varepsilon^{-1}$ does not exist. Consequently, the generalized least squares (GLS) estimator of the regression coefficients in (33) also does not exist. In contrast, when the disturbance covariance matrix is singular, the generalized inverse of the disturbance covariance matrix $\boldsymbol{\Omega}_\varepsilon^{-g}$ does exist. The generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients can be written as:

$$E_g(\boldsymbol{\beta}) = \left(\boldsymbol{X}' \boldsymbol{\Omega}_\varepsilon^{-g} \boldsymbol{X}\right)^{-1} \left(\boldsymbol{X}' \boldsymbol{\Omega}_\varepsilon^{-g} \boldsymbol{y}\right). \tag{34}$$

where $\boldsymbol{\Omega}_\varepsilon^{-g}$ is the generalized inverse of the full covariance matrix of the disturbance terms in (32); and all other terms are the same as described in (33). Thus, if the implicit restrictions are satisfied, the generalized-inverse generalized least squares (GI-GLS) estimator of the regression coefficients in (34) will always exist.

3. Results

3.1. Data

The data consists of a system of seven ($N_C = 7$) currencies, namely, the US dollar (USD), the Eurozone euro (EUR), the Japanese yen (JPY), the Swiss franc (CHF), the British pound (GBP), the Canadian dollar (CAD) and the Australian dollar (AUD). A system of six bilateral exchange rates against the US dollar are sourced from Bloomberg with 49 years of monthly data from Bloomberg from 1st January 1973 to 31st December 2021.

A group of seven local equity market indexes are considered as explanatory variables. The system of seven equity market indexes together with the associated three letter currency code is: United States (USD), Europe (EUR), Japan (JPY), Switzerland (CHF), United Kingdom (GBP), Canada (CAD) and Australia (AUD). The equity market indices are the MSCI total return indices. The German market total return index was used as a proxy for the equity market index associated with the euro.

3.2. Multivariate regression model

The form of a multivariate regression model for the log returns of each of the seven ($N_C = 7$) multilateral exchange rates with two ($N_A = 2$) absolute explanatory variables (including one for the intercept) and one ($N_R = 1$) group of relative explanatory variables can be written as:

$$\Delta \boldsymbol{p}_M^i = \boldsymbol{x}_A^I \alpha_A^{I,i} + \boldsymbol{x}_A^E \beta_A^{E,i} + \boldsymbol{x}_R^{E,i} \beta_R^E + \boldsymbol{\varepsilon}_M^i \tag{35}$$

where $i = 1, \ldots, N_C$; $\Delta \boldsymbol{p}_M^i$ is a $T \times 1$ vector of log returns of the ith multilateral exchange rate; $\boldsymbol{x}_A^I$ is a $T \times 1$ vector of ones; $\alpha_A^{I,i}$ is the ith intercept coefficient; $\boldsymbol{x}_A^E$ is a $T \times 1$ vector of the log returns of the global equity market index; $\beta_A^{E,i}$ is the ith regression coefficient for the log returns of the global equity market index; $\boldsymbol{x}_R^{E,i}$ is a $T \times 1$ vector of the log returns of the ith relative equity market index; β_R^E is the common regression coefficient for the log returns of the group of relative equity market indexes; and $\boldsymbol{\varepsilon}_M^i$ is the ith disturbance term.

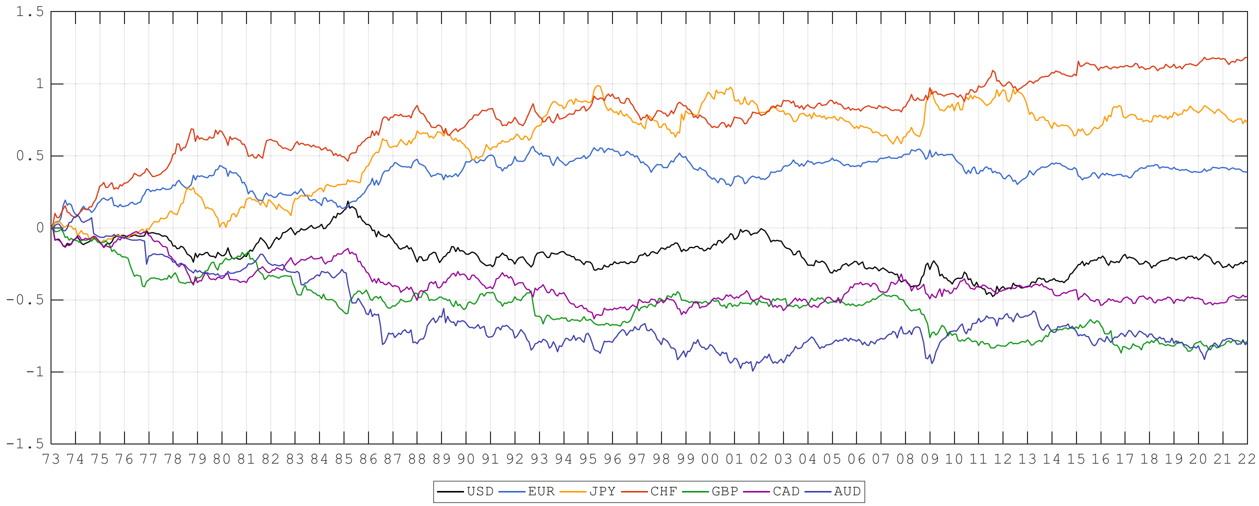

Fig. 1. Cumulative log returns for the multilateral exchange rates. *Notes:* Figure displays the cumulative log returns of the seven multilateral exchange rates.

Table 1
Descriptive statistics for the movements in the multilateral exchange rates.

	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	-0.48 %	6.34 %	-0.003	4.03**	-23.6**	0.046	25.9**
EUR	0.79 %	6.07 %	0.049	4.27**	-23.1**	0.043	40.1**
JPY	1.48 %	8.49 %	0.784**	6.12**	-23.5**	0.041	299.5**
CHF	2.41 %	7.17 %	0.352**	4.81**	-24.5**	0.091	92.9**
GBP	-1.60 %	6.55 %	-0.694**	6.56**	-22.6**	0.063	358.0**
CAD	-0.97 %	6.41 %	-0.252*	4.00**	-24.8**	0.034	30.8**
AUD	-1.63 %	8.58 %	-1.301**	8.38**	-23.6**	0.035	876.8**

Notes: This Table reports the summary statistics for the log returns of each multilateral exchange rate. The summary statistics consist of: annualised average return (Average); the annualised standard deviation (Std. Dev.); skewness (Skew), kurtosis (Kurt), Augmented Dickey-Fuller unit root test (ADF) displaying the test statistic, KPSS test displaying the test statistic; and the Jarque-Bera normality test. Significance levels for skewness, kurtosis, ADF, KPSS, and Jarque-Bera are denoted by * for 5 %, and ** for 1 %.

Table 2
Correlation matrix.

	USD	EUR	JPY	CHF	GBP	CAD	AUD
USD	1.000	-0.521**	-0.053	-0.496**	-0.165**	0.439**	-0.105*
EUR	-0.521**	1.000	-0.174**	0.626**	0.035	-0.471**	-0.348**
JPY	-0.053	-0.174**	1.000	0.012	-0.263**	-0.376**	-0.356**
CHF	-0.496**	0.626**	0.012	1.000	-0.062	-0.605**	-0.423**
GBP	-0.165**	0.035	-0.263**	-0.062	1.000	-0.172**	-0.226**
CAD	0.439**	-0.471**	-0.376**	-0.605**	-0.172**	1.000	0.271**
AUD	-0.105*	-0.348**	-0.356**	-0.423**	-0.226**	0.271**	1.000

Notes: This Table reports the correlation matrix for the log returns of the multilateral exchange rates. Significance levels for the correlations are denoted by * for 5 %, and ** for 1 %.

3.3. Dependent variables

The dependent variables consist of the log returns of the seven ($N_C = 7$) multilateral exchange rates. The log returns of the N_C multilateral exchange rates are priced in terms of an equally-weighted basket of all N_C numéraire currencies in (6). Note that all N_C currencies are numéraire currencies of the multicurrency numéraire. For example, the US dollar multilateral exchange rate is calculated using (6) to give:

$$\Delta p_{M,t}^{USD} = \frac{1}{7}\sum_{j=1}^{7}\Delta p_{B,t}^{USD/j} = \frac{1}{7}\Delta p_{B,t}^{USD/USD} + \frac{1}{7}\Delta p_{B,t}^{USD/EUR} + \frac{1}{7}\Delta p_{B,t}^{USD/JPY} + \frac{1}{7}\Delta p_{B,t}^{USD/CHF} + \frac{1}{7}\Delta p_{B,t}^{USD/GBP} + \frac{1}{7}\Delta p_{B,t}^{USD/CAD} + \frac{1}{7}\Delta p_{B,t}^{USD/AUD} \tag{36}$$

where $t = 1, \ldots, T$; $\Delta p_{M,t}^{USD}$ is the log return of the US dollar multilateral exchange rate; and $\Delta p_{B,t}^{USD/j}$ is the log return of the USD/jth bilateral exchange rate with $\Delta p_{B,t}^{USD/USD} = 0$ and $j = 1, \ldots, N_C$.

Fig. 1 displays the cumulative log returns of each multilateral exchange rate. Table 1 reports the descriptive statistics for the log returns of the system of seven multilateral exchange rates. The Swiss franc (CHF) has the highest average annualised return of 2.41%, and the Australian dollar has the lowest average annualised return of -1.63%, which is closely followed by the British pound (GBP) with an average annualised return of -1.60%. The highest annualised volatilities of 8.58% and 8.49% are associated with the Australian dollar (AUD) and the Japanese yen (JPY), respectively. In contrast, the lowest annualised volatility of 6.07 % is associated with the Eurozone euro (EUR).

Significant positive skewness values of 0.784 and 0.352 are associated with the safe-haven currencies of the Japanese yen (JPY) and the Swiss franc (CHF), respectively. In contrast, significant negative skewness values of -1.301, -0.694, and -0.252 are associated with the Australian dollar (AUD), the British pound (GBP), and the Canadian dollar (CAD), respectively. The movements in all multilateral exchange rates experience significant ADF statistics (Dickey & Fuller, 1979) and insignificant KPSS statistics (Kwiatkowski et al., 1992), which provide significant evidence of stationarity. The movements in all of the multilateral exchange rates also experience significant kurtosis and significant Jarque-Bera statistics (Jarque & Bera, 1987), which indicate non-normal log returns.

Table 2 reports the correlation matrix for the log returns of the multilateral exchange rates. There are three highly significant positive observed correlations. The first positive observed correlation is 0.626 between the Eurozone euro (EUR) and the Swiss franc (CHF), which is driven by the geographically close proximity of the Eurozone and Switzerland. The second positive observed correlation is 0.439 between the US dollar (USD) and the Canadian dollar (CAD), which is driven by the geographically close proximity of the US and Canada. The third positive observed correlation is 0.271 between the Canadian dollar (CAD) and the Australian dollar (AUD), which is driven by both currencies being associated with commodity exporting countries: commodity currencies (Chen & Rogoff, 2003).

Table 3
Descriptive statistics for the explanatory variables.

Panel A: Hybrid explanatory variables							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	10.27%	15.40 %	-0.666**	5.53**	-22.7**	0.113	200.1**
EUR	7.83 %	19.62 %	-0.852**	6.01**	-22.5**	0.043	294.1**
JPY	4.89 %	18.07 %	-0.393**	4.51**	-22.1**	0.133	71.1**
CHF	7.94 %	15.76 %	-0.775**	6.50**	-21.3**	0.122	360.0**
GBP	9.86 %	18.33 %	0.262**	12.67**	-22.0**	0.057	2301.5**
CAD	8.88 %	16.20 %	-0.853**	6.62**	-22.3**	0.038	392.9**
AUD	10.10 %	18.88 %	-1.967**	20.61**	-23.1**	0.053	7989.0**
AVG	**8.54 %**	**17.47 %**	**-0.749**	**8.92**			
Panel B: Absolute explanatory variable							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
GBL	8.54 %	13.65 %	-1.312**	9.39**	-20.7**	0.083	1169.6**
Panel C: Relative explanatory variables							
	Average	Std. Dev.	Skew	Kurt	ADF	KPSS	Jarque-Bera
USD	1.73 %	7.90 %	0.469**	4.23**	-27.5**	0.069	58.5**
EUR	-0.71 %	12.13 %	-0.435**	5.47**	-24.9**	0.027	168.2**
JPY	-3.65 %	13.81 %	0.057	4.43**	-24.9**	0.144	50.6**
CHF	-0.60 %	9.11 %	0.001	3.84**	-23.7**	0.114	17.3**
GBP	1.32 %	10.94 %	1.105**	13.44**	-23.5**	0.016	2792.3**
CAD	0.34 %	9.56 %	-0.087	3.54**	-24.9**	0.061	8.0*
AUD	1.56 %	12.43 %	-0.313**	7.83**	-27.1**	0.028	581.8**
AVG	**0.00 %**	**10.84 %**	**0.114**	**6.11**			

Notes: This Table reports the summary statistics for the explanatory variables, with Panel A reporting the group of hybrid explanatory variables (one for each currency), Panel B reporting the absolute explanatory variable and Panel C reporting the group of relative explanatory variables (one for each currency). The summary statistics consist of: annualised average return (Average); the annualised standard deviation (Std. Dev.); skewness (Skew), kurtosis (Kurt), Augmented Dickey-Fuller unit root test (ADF) displaying the test statistic, KPSS test displaying the test statistic; and the Jarque-Bera normality test. Significance levels for skewness, kurtosis, ADF, KPSS, and Jarque-Bera are denoted by * for 5 %, and ** for 1 %.

3.4. Explanatory variables

The explanatory variables consist of the movements in a group of local equity market indexes. There is one local equity market index associated with each currency. However, the cross-sectional sum of the movements in the group of local equity market indexes does not equal to zero. Thus, the movements in a group of local equity market indexes are classified as a group of hybrid explanatory variables. The decomposition of a group of hybrid explanatory variables in (23) is used to decompose the log return of the *i*th local equity market indexes at time *t* to give:

$$x_{H,t}^{E,i} = x_{A,t}^{E} + x_{R,t}^{E,i} \tag{37}$$

where $i = 1,\ldots,N_C$; $t = 1,\ldots,T$; $x_{H,t}^{E,i}$ is the log returns of the *i*th equity market associated with the *i*th currency; $x_{A,t}^{E} = \frac{1}{N_C}\sum_{i=1}^{N_C} x_{H,t}^{E,i}$ is the log return of the global equity market index (absolute explanatory variable) using (24); and $x_{R,t}^{E,i} = x_{H,t}^{E,i} - x_{A,t}^{E}$ is the log return of the *i*th relative equity market index (relative explanatory variable) using (25).

Table 3 reports the descriptive statistics for the group of hybrid explanatory variables (Panel A), together with the absolute explanatory variable (Panel B) and the group of relative explanatory variables (Panel C). The absolute explanatory variable (Panel B) represents the movements in a global equity market index (GBL) and calculated by the cross-sectional average of the log returns for the group of local equity market indexes. The group of relative explanatory variables (one for each currency) are calculated from the differences between the movements in the local equity market indexes (Panel A) and the movements in the global equity market index (Panel B).

The movements in the global equity market index (absolute explanatory variable) have an annualised return of 8.54 %, and an annualised volatility of 13.65 %. In contrast, the group of the movements in the relative equity market index (relative explanatory variables) have an average annualised return of exactly 0.00 % by construction, and an average annualised volatility of 10.84 %. The movements in the relative US equity market index (USD) outperform the movements in the global equity market the most by 1.73 % annually. In contrast, movements in the relative Japanese equity market index (JPY) underperform the movements in the global equity market the most by − 3.65 % annually.

All of the explanatory variables experience significant ADF statistics (Dickey & Fuller, 1979) and insignificant KPSS statistics (Kwiatkowski et al., 1992), which provide significant evidence of stationarity. All of the explanatory variables also experience significant kurtosis and significant Jarque-Bera statistics (Jarque & Bera, 1987), which indicate non-normal log returns.

Table 4
Multivariate regression results.

i	$\alpha_A^{I,i}$	$\beta_A^{E,i}$	β_R^E
USD	-0.0004	-0.0610**	-0.0247**
EUR	0.0007	-0.0383**	-0.0247**
JPY	0.0012*	-0.1375**	-0.0247**
CHF	0.0020**	-0.1174**	-0.0247**
GBP	-0.0013**	0.0159	-0.0247**
CAD	-0.0008*	0.1193**	-0.0247**
AUD	-0.0014*	0.2191**	-0.0247**
AVG	0.0000	0.0000	

Notes: This Table reports the observed regression coefficients from the multivariate regression model in (35). Significance levels for the regression coefficients are denoted by * for 5 %, and ** for 1 %.

3.5. Multivariate regression output

Table 4 reports the observed regression coefficients from the multivariate regression model in (35) for the entire 49-year data sample. The regression coefficients are estimated using the generalized-inverse generalized least squares (GI-GLS) estimator in (34). The first data column of Table 4 reports the intercept coefficients ($\alpha_A^{I,i}$) for the movements in the seven multilateral exchange rates. The Swiss franc (CHF) and the Japanese yen (JPY) have significant positive intercept coefficients of 0.0020 and 0.0012, respectively. In contrast, the Australian dollar (AUD), the British pound (GBP), and the Canadian dollar (CAD) have significant negative intercept coefficients of -0.0014, -0.0013 and -0.0008, respectively.

The second data column of Table 4 reports regression coefficients ($\beta_A^{E,i}$) associated with the movements in the global equity market index (absolute explanatory variable). The Australian dollar (AUD) and the Canadian dollar (CAD) move significantly with the global equity market index, with observed regression coefficients of 0.2191 and 0.1193, respectively. In contrast, the Japanese yen (JPY), the Swiss franc (CHF), the US dollar (USD), and the Eurozone euro (EUR) move significantly against the global equity market index, with observed regression coefficients of -0.1375, -0.1174, -0.0610, and -0.0383, respectively.

The third data column of Table 4 reports the common regression coefficient (β_R^E) associated with the movements in the group of relative equity market indexes (a group of relative explanatory variables). The common regression coefficient is the same for all currencies and is simply repeated for each currency. Currencies move significantly against the relative equity market indexes with a value of -0.0247. This shows that the currencies are shock-absorbers, so that when the local equity market index out (under) performs relative to the global equity market index, the associated currency under (out) performs.

4. Conclusion

A multicurrency numéraire is a weighted basket of numéraire currencies. A no-arbitrage condition enforces the movements in a system of multilateral exchange rates associated with the numéraire currencies of the multicurrency numéraire to be a singular system. Singular systems pose a methodological challenge in a multivariate regression model. This paper proposed a solution to overcome the methodological challenge that consisted of two parts. The first part imposed implicit restrictions on the explanatory variables and the regression coefficients to guarantee that the disturbance covariance matrix was singular. The second part replaced the ordinary inverse with the generalized inverse to modify the generalized least squares (GLS) estimator of the regression coefficients (see Theil, 1971).

The solution provided a consistent multivariate regression framework to explain the observed heterogeneity in the relative currency market. For example, all currencies cannot move with, or against, absolute explanatory variables. In this situation, the heterogeneity appears in the observed regression coefficients. In addition, all currencies move in a common (panel-like) fashion with, or against, a group of relative explanatory variables. In this situation, the heterogeneity appears in the currency-specific explanatory variables.

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References

Barten, A. P. (1969). Maximum likelihood estimation of a complete system of demand equations. *European Economic Review, 1*(1), 7–73.
Bewley, R. (1986). *Allocation models: Specification, estimation and applications*. Cambridge, MA: Ballinger.
Campbell, J. Y., Medeiros, K. S., & Viceira, L. M. (2010). Global currency hedging. *Journal of Finance, 65*, 87–121.
Chacholiades, M. (1971). The sufficiency of three-point arbitrage to ensure consistent cross rates of exchange. *Southern Economic Journal, 38*, 86–88.
Chen, Y., & Rogoff, K. S. (2003). Commodity currencies. *Journal of International Economics, 60*(1), 133–160.
Dickey, D. A., & Fuller, W. A. (1979). Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association, 74*, 427–431.
Dhrymes, P. J., & Schwarz, S. (1987a). On the existence of generalized inverse estimators in a singular system of equations. *Journal of Forecasting, 6*, 181–192.
Dhrymes, P. J., & Schwarz, S. (1987b). On the invariance of estimators for singular systems of equations. *Greek Economic Review, 9*(1), 231–250.
Engel, C., & West, K. D. (2005). Exchange rates and fundamentals. *Journal of Political Economy, 113*(3), 485–517.

Fiebig, D. G. (2001). Seemingly unrelated regression. In B. H. Baltagi (Ed.), *A companion to theoretical econometrics* (pp. 101–121). Blackwell.
Frankel, J., & Wei, S. (1994). Yen bloc or dollar bloc? Exchange rate policies in East Asian economies. In T. Ito, & A. O. Krueger (Eds.), *Macroeconomic linkages: Savings, exchange rates, and capital flows* (pp. 295–329). Chicago: University of Chicago Press.
Haynes, S. E., & Stone, J. A. (1994). The relationship between bilateral and multilateral models of exchange rates. *Journal of International Money and Finance, 13*(1), 71–82.
Hovanov, N. V., Kolari, J. W., Mikhail, T., & Sokolov, V. (2004). Computing currency invariant indexes with an application to minimum variance currency collections. *Journal of Economic Dynamics and Control, 28*, 1481–1504.
Haupt, H., & Oberhofer, W. (2002). Fully restricted linear regression: A pedagogical note. *EconomicsBulletin, 3*(1), 1–7.
Haupt, H., & Oberhofer, W. (2006). Generalized adding-up in systems of regression equations. *Economics Letters, 92*(2), 263–269.
Jarque, C. M., & Bera, A. K. (1987). A test for normality of observations and regression residuals. *International Statistical Review, 55*(2), 163–172.
Kreijger, R. G., & Neudecker, H. (1977). Exact linear restrictions on parameters in the general linear model with a singular covariance matrix. *Journal of the American Statistical Association, 72*, 430–432.
Kunkler, M. (2021). The Chinese renminbi's co-movement with the US dollar: Addressing the numéraire issue. *Finance Research Letters, 40*, Article 101741.
Kunkler, M., & MacDonald, R. (2015). Half-lives of currencies and aggregation bias. *Economics Letters, 135*, 58–60.
Kwiatkowski, D., Phillips, P. C. B., Schmidt, P., & Shin, Y. (1992). Testing the null hypothesis of stationarity against the alternative of a unit root. *Journal of Econometrics, 54*, 159–178.
Meese, R. A., & Rogoff, K. S. (1983a). Empirical exchange rate models of the seventies: Do they fit out of sample? *Journal of International Economics, 14*(1–2), 3–24.
Meese, R. A., & Rogoff, K. S. (1983b). *The out of sample failure of empirical exchange models. in exchange rates and international macroeconomics*. In A. Jacob Frenkel. (Ed.). Chicago: Univ. Chicago Press.
Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory, 13*, 341–360.
Siegel, J. J. (1972). Risk, interest rate and the forward market. *Quarterly Journal of Economics, 86*, 303–309.
Theil, H. (1971). *Principles of econometrics*. New York: Wiley,.