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Abstract. Inspired by our previous work on the boundedness of Toeplitz operators, we introduce weak BMO and VMO type conditions, denoted by BWMO and VWMO, respectively, for functions on the open unit disc of the complex plane. We show that the average function of a function $f \in$ BWMO is boundedly oscillating, and the analogous result holds for $f \in$ VWMO. The result is applied for generalizations of known results on the essential spectra and norms of Toeplitz operators. Finally, we provide examples of functions satisfying the VWMO condition which are not in the classical VMO or even in BMO.

Contents

1. Introduction and main results

Consider the Banach space $L^p := (L^p(\mathbb{D}, dA), \|\cdot\|_p)$, where $1 < p < \infty$ and dA is the normalized area measure on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, and the *Bergman space* A^p, which is the closed subspace of L^p consisting of analytic functions. The *Bergman projection* P is the orthogonal projection of L^2 onto A^2, and it has the integral representation

$$Pg(z) = \int_{\mathbb{D}} K_\zeta(z) g(\zeta) dA(\zeta)\,, \quad \text{where } K_\zeta(z) = \frac{1}{(1-z\bar{\zeta})^2},\ z, \zeta \in \mathbb{D},$$

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is the *Bergman kernel*. It is also known to be a bounded projection of L^p onto A^p for every $1 < p < \infty$. For an integrable function $f : \mathbb{D} \to \mathbb{C}$ and, say, bounded analytic functions g, the *Toeplitz operator* T_f with *symbol* f is defined by

$$T_f g = P(fg) = \int_{\mathbb{D}} \frac{f(\zeta)g(\zeta)}{(1 - z\bar{\zeta})^2} dA(\zeta). \tag{1.1}$$

A related class of operators consists of *Hankel operators* $H_f : A^p \to L^p$ defined by

$$H_f g = (I - P)(fg),$$

where I is the identity operator and $I - P$ is the complementary projection of P. Notice that the boundedness of P implies that both $T_f : A^p \to A^p$ and $H_f : A^p \to L^p$ are bounded whenever the symbol f is in L^∞.

In [8] and [9], we studied a generalized definition of Toeplitz operators with locally integrable symbols f satisfying a weak "averaging" condition (see (3.4) below), and showed that one can define $T_f = \lim_{\rho \to 1} T_{f_\rho}$, where $f_\rho = \chi_\rho f$ and χ_ρ is the characteristic function of the compact set $\{z \in \mathbb{D} : |z| \le \rho\}$, $\rho < 1$. The limit converges in the strong operator topology and the generalized definition coincides with (1.1), whenever the latter makes sense. It was recently shown in [10] that the condition is however not necessary for the boundedness of T_f.

Here, our aim is to apply the same idea to introduce new *weak* BMO and VMO type conditions BWMO and VWMO: we replace the standard definition of BMO (and VMO) by the above described weak averaging condition. It is quite clear there are functions which belong to VWMO but are not in VMO or not even in BMO. We will exhibit concrete examples in Section 4, see Example 4.8. Our new definition leads to the following results. First, we prove that whenever f satisfies the BWMO (or VWMO) condition, then the average function $\widehat{f}$ belongs to the space BO (or in VO) of functions of bounded (or vanishing) oscillation—see Theorem 3.8. This allows us to extend the standard results on the essential spectra and Fredholm properties of Toeplitz operators T_f from the case $f \in L^\infty \cap \text{VMO}$ (see [1, 5] and the references therein) to integrable, not necessarily bounded, symbols in VWMO.

The weak conditions are broader in scope, and should have more applications, which we hope to demonstrate in future work on Toeplitz and related operators.

2. Preliminaries

In this section, we explain the notions used in the paper and recall a number of basic results that we need in the subsequent sections.

The space of bounded mean oscillation BMO^p provides a class of symbols f, strictly larger than L^∞, for which bounded (or compact) Toeplitz operators can be characterized in terms of the boundary behavior of the Berezin transform

$\widetilde{f}$. Similarly, its closed subspace of vanishing mean oscillation VMO^p plays an important role in the study of other (spectral) properties of Toeplitz operators. Let $r > 0$, $1 \le p < \infty$ and f be a locally L^p-integrable function on $\mathbb{D}$. We say that f is of *bounded mean oscillation*, and write $f \in \mathrm{BMO}^p_r$, if

$$\sup_{z\in\mathbb{D}} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(\zeta) - \widehat{f}_r(z)|^p \, dA(\zeta) < \infty, \tag{2.1}$$

where $|B| = \int_B dA$ for any measurable set $B \subset \mathbb{D}$, and $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$ is the disc with center z and radius r in the Bergman metric $\beta : \mathbb{D} \times \mathbb{D} \to \mathbb{R}^+$. Moreover, $\widehat{f}_r$ is the average function defined by

$$\widehat{f}_r(z) = \frac{1}{|D(z,r)|} \int_{D(z,r)} f \, dA \tag{2.2}$$

for $z \in \mathbb{D}$. If, in addition,

$$\lim_{|z|\to 1} \frac{1}{|D(z,r)|} \int_{D(z,r)} |f(\zeta) - \widehat{f}_r(z)|^p \, dA(\zeta) = 0,$$

we say that f is in VMO^p_r. These definitions are independent of r, and we write simply BMO^p and VMO^p for the two spaces when $r = 1$.

To decompose BMO^p and VMO^p into smaller spaces, we define the *oscillation* $\omega(f)$ of a continuous function f by

$$\omega(f)(z) = \sup_{w\in D(z,1)} |f(z) - f(w)| \tag{2.3}$$

for $z \in \mathbb{D}$. (We fix here the radius of the hyperbolic disc in order to keep the notation simple.) The space of *bounded oscillation* BO consists of all continuous functions f for which $\omega(f) \in L^\infty$. We say $f \in \mathrm{BO}$ is of *vanishing oscillation* and write $f \in \mathrm{VO}$ if $\omega(f)(z) \to 0$ as $|z| \to 1$. The spaces BA^p and VA^p of functions f of *bounded* or *vanishing average* are defined by requiring that $\widehat{|f|^p_1} \in L^\infty$ or $\widehat{|f|^p_1}(z) \to 0$ as $|z| \to 1$, respectively. These spaces provide the useful decompositions

$$\mathrm{BMO}^p = \mathrm{BO} + \mathrm{BA}^p \quad \text{and} \quad \mathrm{VMO}^p = \mathrm{VO} + \mathrm{VA}^p \tag{2.4}$$

for $p \ge 1$, which can be obtained by setting $f = \widehat{f}_1 + (f - \widehat{f}_1)$. For the proofs and further details, see [12].

We will also need the Berezin transform $\widetilde{f}$ of f, which plays an important role in characterizations of various properties of Toeplitz operators. Given $f \in L^1$, the Berezin transform is defined by setting (see [13], Sect. 6.3.)

$$\widetilde{f}(z) = \frac{\langle fK_z, K_z\rangle_{L^2}}{\langle K_z, K_z\rangle_{L^2}} = \int_{\mathbb{D}} f|k_z|^2 \, dA = \int_{\mathbb{D}} (f \circ \phi_z)(\zeta) \, dA(\zeta), \tag{2.5}$$

where $k_z = K_z \|K_z\|_2^{-1}$ and $\phi_z(w) = \frac{z-w}{1-w\bar{z}}$ is the Möbius transform interchanging 0 and z. It is a direct consequence of the definition that the Berezin transform of any function $f \in L^\infty$ is bounded and continuous.

Recall that a bounded linear operator T on a Banach space X is called a *Fredholm* operator, if its kernel $\ker T$ and cokernel are both finite dimensional. If T is Fredholm on X, its *index* is defined as the number $\operatorname{ind} T = \dim \ker T - \dim(X/T(X))$. The *essential spectrum* $\operatorname{spec}_{\mathrm{ess}}(T) = \operatorname{spec}_{\mathrm{ess}}(T : X \to X)$ of T is defined by

$$\operatorname{spec}_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

which is clearly contained in the spectrum $\operatorname{spec}(T) = \operatorname{spec}(T : X \to X)$ of T. Similarly, the *essential norm* is the expression $\|T\|_{\mathrm{ess}} = \inf_K \|T + K\|$, where on the right we have the operator norm of $T + K$ and the infimum is taken over all compact operators $K : X \to X$. Clearly, $\|T\|_{\mathrm{ess}} \le \|T\|$.

We still need one more notion to formulate the next theorem. The *Stone-Čech compactification* $\beta\mathbb{D}$ of $\mathbb{D}$ is defined by its universal property that any continuous map f from $\mathbb{D}$ to a compact Hausdorff space K can be uniquely extended to a continuous map $f : \beta\mathbb{D} \to K$. Here, we do not distinguish between f and its extension to $\beta\mathbb{D}$. Note that $\beta\mathbb{D}$ can be realized as the maximal ideal space of bounded continuous functions defined on $\mathbb{D}$. Every maximal ideal corresponds to a point in $\beta\mathbb{D}$ via evaluation. See e.g. [6] for the use of $\beta\mathbb{D}$ in the topic under consideration.

The following result is known and our aim is to extend it for a larger symbol class, see Section 4.

Theorem 2.1. *Let* $1 < p < \infty$ *and* $f \in L^\infty \cap \mathrm{VMO}^1$.

(i) *We have*

$$\operatorname{spec}_{\mathrm{ess}}(T_f : A^p \to A^p) = \bigcup_{y \in \beta\mathbb{D} \setminus \mathbb{D}} \widetilde{f}(y) = \widetilde{f}(\beta\mathbb{D} \setminus \mathbb{D}), \tag{2.6}$$

where $\widetilde{f}$ *denotes the extension of the Berezin transform of* f *to the Stone-Čech compactification* $\beta\mathbb{D}$ *of* $\mathbb{D}$.

(ii) *If* T_f *is Fredholm on* A^p*, then the index of* T_f *equals the negative winding of* $\widetilde{f}|_{\{|z|=r\}}$*, where* r *is sufficiently close to* 1.

Formula (2.6) was obtained for the classical Bergman space $A^2(\mathbb{D})$ in [11]. For arbitrary $1 < p < \infty$, (2.6) was proved in [5] using elementary methods and in [1, 2] using techniques with band-dominated operators. The index formula stated in (ii) was proved in [6] for the Hilbert space A^2 and can be treated analogously for other values of p. We remark that, as shown in [6], formula (2.6) and claim (ii) also hold for Toeplitz operators $T_f : A^2 \to A^2$ with symbols in a larger algebra $\mathcal{A}$ consisting of bounded functions f such that $H_f : A^2 \to L^2$ is compact (see Section 5 and Theorems 19, 24 of the citation). Our generalization is formulated in Corollary 4.2, and it involves also unbounded symbols among other things.

3. Weak *BMO*-type conditions on the unit disc

In this section we introduce the weak BMO-type condition which is interesting in itself and may be applied to other considerations as well.

Definition 3.1. For all $z = re^{i\theta} \in \mathbb{D}$ with $r \in [0, 1)$ and $\theta \in [0, 2\pi)$ we denote

$$B(z) = \{\rho e^{i\phi} \mid r \le \rho \le 1 - \frac{1}{2}(1-r)\,,\ \theta \le \phi \le \theta + \pi(1-r)\}. \tag{3.1}$$

We denote, for $\zeta = \tilde{r}e^{i\tilde{\theta}} \in B(z)$,

$$B(z, \zeta) = \{\rho e^{i\phi} \mid r \le \rho \le \tilde{r}\,,\ \theta \le \phi \le \tilde{\theta}\} \tag{3.2}$$

and

$$\widehat{f}(z, \zeta) := \frac{1}{|B(z)|} \int\limits_{B(z,\zeta)} f\, dA, \tag{3.3}$$

where $f \in L^1_{\mathrm{loc}}$.

Remark 3.2. (i) In (3.1) it may of course happen that $\theta + \pi(1-r) > 2\pi$. This does not harm the definition of the set $B(z)$, but in (3.2) and (3.3) and in the sequel in all similar places we must agree that the relation $\zeta = \tilde{r}e^{i\tilde{\theta}} \in B(z)$ is understood to imply $\tilde{\theta} \in [\theta, \theta + \pi(1-r)]$, even if $\theta + \pi(1-r) > 2\pi$.

Using this convention, we define for $\zeta_j = \rho_j e^{i\phi_j} \in B(z)$, $j = 1, 2$, the notion $\zeta_1 \precsim \zeta_2$, if $\rho_1 \le \rho_2$ and $\phi_1 \le \phi_2$.

(ii) In [8] and [9] we specified a sequence $(z_n)_{n=1}^{\infty} \subset \mathbb{D}$ such that the corresponding sets $B_n := B(z_n)$ form an essentially disjoint union of the disc $\mathbb{D}$. Here, sets are called essentially disjoint, if they are disjoint save possibly their boundaries. We will use this decomposition later.

In the following we will study symbols f for which there exists a constant $C > 0$ such that

$$|\widehat{f}(z, \zeta)| \le C \tag{3.4}$$

for all $z \in \mathbb{D}$ and all $\zeta \in B(z)$. By Theorem 2.3 in [8], if (3.4) holds for the symbol f, the Toeplitz operator $T_f : A^p \to A^p$, defined as the limit

$$\lim_{\rho \to 1^-} T_{\chi_\rho f} \tag{3.5}$$

converging in the strong operator topology, is bounded. Recall that χ_ρ denotes the characteristic function of the set $\mathbb{D}_\rho = \{w \in \mathbb{D} : |w| \le \rho\}$ with $\rho < 1$. Also, according to [8], Theorem 2.5, if $f \in L^1_{\mathrm{loc}}$ is such that in addition to (3.4) there holds

$$\lim_{|z| \to 1} \frac{1}{|B(z)|} \sup_{\zeta \in B(z)} \Big| \int\limits_{B(z,\zeta)} f\, dA \Big| = 0, \tag{3.6}$$

then $T_f : A^p \to A^p$ is compact.

Given a precompact subset $K \subset \mathbb{D}$ with $|K| > 0$, we denote the average of f in K by (cf. (2.2))

$$\widehat{f}_K = \frac{1}{|K|} \int_K f \, dA. \tag{3.7}$$

For all $f \in L^1_{\mathrm{loc}}$ we also define the average function $\widehat{f} : \mathbb{D} \to \mathbb{C}$ by

$$\widehat{f}(z) = \widehat{f}_{B(z)} \,, \quad z \in \mathbb{D}. \tag{3.8}$$

Then, we have $\widehat{f} \in C(\mathbb{D})$ (the space of continuous functions on the open disc).

Remark 3.3. Note that the more standard definition of the average function using a hyperbolic disc instead of $B(z)$ was already introduced in Section 2, and we keep the difference in the notation, $\widehat{f}$ vs. $\widehat{f}_r$, to indicate this. We will need the present definition of $\widehat{f}$ for technical reasons. Moreover, it is easy to see that in the definitions of the spaces BO and VO one can replace the sets $D(z,1)$ by the sets $B(z)$ without changing the concept. This follows from the simple geometric observation that there exists a number $N \in \mathbb{N}$ such that for all $z \in \mathbb{D}$, the set $D(z,1)$ is contained in the union of at most N sets $B(w)$ and conversely, $B(z)$ is contained in the union of at most N sets $D(w,1)$. For similar reasons, in the definition of the spaces BA^p and VA^p the average functions $\widehat{|f|^p_1}$ could be replaced by the average functions $\widehat{|f|^p}$. We leave the details of the proofs for these claims to the reader.

Definition 3.4. Let us consider functions $f \in L^1_{\mathrm{loc}}$ and define the following BMO-type condition

$$\|f\|_{\mathrm{BWMO}} := \sup_{z \in \mathbb{D}} \frac{1}{|B(z)|} \sup_{\zeta \in B(z)} \Big| \int_{B(z,\zeta)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi) \Big| < \infty. \tag{3.9}$$

We refer to this definition as the BWMO-condition (for "bounded weak mean oscillation," not to be confused with the existing term of "weak BMO" in the literature).

We also introduce the following VWMO-condition (for vanishing weak mean oscillation) for a function $f \in L^1_{\mathrm{loc}}$,

$$\lim_{|z| \to 1} \frac{1}{|B(z)|} \sup_{\zeta \in B(z)} \Big| \int_{B(z,\zeta)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi) \Big| = 0. \tag{3.10}$$

It is easy to see that the expression $\|\cdot\|_{\mathrm{BWMO}}$ is a seminorm for example in the space of bounded continuous functions in $\mathbb{D}$ and that $\|f\|_{\mathrm{BWMO}} = 0$, if and only if f is constant. To see the latter statement, if f is a non-constant, bounded and continuous function in $\mathbb{D}$, we pick up a point $z \in \mathbb{D}$ such that, for a neighborhood U of z, the function f is non-constant in $U \cap B(z)$. Then, it is clear that the expression

$$\int_{B(z,\zeta)} \big(f - \widehat{f}(z)\big)\, dA$$

cannot be constant for $\zeta \in U \cap B(z)$, which implies that $\|f\|_{\mathrm{BWMO}} \neq 0$.

We will use the following fact.

Lemma 3.5. *Assume* (3.9) *holds for a function $f \in L^1_{\mathrm{loc}}(\mathbb{D})$. Let $z \in \mathbb{D}$ be arbitrary and assume that the points $\tilde{z}, \zeta$, $\tilde{z} \precsim \zeta$, belong to $B(z)$ (thus, $B(\tilde{z}, \zeta) \subset B(z)$). Then, we have*

$$\frac{1}{|B(z)|}\Big| \int_{B(\tilde{z},\zeta)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi)\Big| \le C\|f\|_{\mathrm{BWMO}} \tag{3.11}$$

for some constant $C > 0$.

Proof. We start by the following elementary geometric observation: if $g \in L^1_{\mathrm{loc}}(\mathbb{D})$, $z = re^{i\theta} \in \mathbb{D}$ and $\zeta_1, \zeta_2 \in B(z)$ are such that $z \precsim \zeta_1 \precsim \zeta_2$ then the integral over the set $B(\zeta_1, \zeta_2)$ can be presented as

$$\int_{B(\zeta_1,\zeta_2)} g\, dA = \sum_{j=1}^{4} \gamma_j \int_{B(z,w_j)} g\, dA, \tag{3.12}$$

where $\gamma_j \in \{-1, 1\}$ and w_j, $j = 1, \dots, 4$ are some points in $B(z)$ with $z \precsim w_j \precsim \zeta_2$. Indeed, if $\zeta_j = \rho_j e^{i\phi_j}$, $j = 1, 2$ we choose

$$w_1 = \zeta_2,\ w_2 = \rho_1 e^{i\phi_2},\ w_3 = \rho_2 e^{i\phi_1},\ w_4 = \zeta_1. \tag{3.13}$$

Then, we have

$$B(\zeta_1, \zeta_2) = B(z, w_1) \setminus \Big(B(z, w_3) \cup \big(B(z, w_2) \setminus B(z, w_4)\big)\Big)$$

and formula (3.12) follows from this, since the sets $B(z, w_3)$ and $B(z, w_2) \setminus B(z, w_4)$ are essentially disjoint.

We now apply formula (3.12) to the integral in (3.11) and obtain

$$\int_{B(\tilde{z},\zeta)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi) = \sum_{j=1}^{4} \gamma_j \int_{B(z,w_j)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi) \tag{3.14}$$

for some points $w_j \in B(z)$. The bound (3.11) follows from this and the triangle inequality, since (3.9) implies

$$\frac{1}{|B(z)|}\Big| \int_{B(z,w_j)} \big(f(\xi) - \widehat{f}(z)\big) dA(\xi)\Big| \le \|f\|_{\mathrm{BWMO}} \tag{3.15}$$

for all j. □

Remark 3.6. If, in addition, (3.10) holds for $f \in L^1_{\rm loc}$, then

$$\lim_{|z|\to 1} \frac{1}{|B(z)|} \sup_{\substack{\tilde z,\zeta \in B(z) \\ \tilde z \precsim \zeta}} \Big| \int\limits_{B(\tilde z,\zeta)} (f(\xi) - \widehat{f}(z)) dA(\xi) \Big| = 0. \tag{3.16}$$

This is so since the assumption (3.10) implies that in (3.14), (3.15) an arbitrarily small multiplier $\varepsilon > 0$ can be added to the right hand side, if $|z|$ is close enough to 1. Following the proof, also the right hand side of (3.15) then can be multiplied with ε, which proves the claim.

Corollary 3.7. *There is a constant $C > 0$ such that, if $f \in L^1_{\rm loc}$ satisfies (3.9), then we have, for all $z \in \mathbb{D}$ and $\tilde z, \zeta \in B(z)$ with $\tilde z \precsim \zeta$,*

$$\Big| \widehat{f}(z) - \widehat{f}_{B(\tilde z,\zeta)} \Big| \le C \|f\|_{\rm BWMO}, \tag{3.17}$$

provided that the points $z, \tilde z, \zeta$ in addition satisfy

$$\frac{|B(z)|}{|B(\tilde z, \zeta)|} \le 2. \tag{3.18}$$

If in addition (3.10) holds for $f \in L^1_{\rm loc}$, then we have

$$\lim_{|z|\to 1} \sup_{\tilde z,\zeta} \Big| \widehat{f}(z) - \widehat{f}_{B(\tilde z,\zeta)} \Big| = 0, \tag{3.19}$$

where the supremum is taken over all $\tilde z, \zeta \in B(z)$ with $\tilde z \precsim \zeta$ satisfying also (3.18).

Proof. To prove the first statement, we fix $z \in \mathbb{D}$ and using Lemma 3.5 obtain for the average of f over the set $B(\tilde z, \zeta)$

$$\begin{aligned} |\widehat{f}_{B(\tilde z,\zeta)} - \widehat{f}(z)| &= \frac{1}{|B(\tilde z,\zeta)|} \Big| \int\limits_{B(\tilde z,\zeta)} (f(\xi) - \widehat{f}(z)\, dA(\xi) \Big| \\ \le\ & C \|f\|_{\rm BWMO} \frac{|B(z)|}{|B(\tilde z,\zeta)|} \le C' \|f\|_{\rm BWMO}, \end{aligned} \tag{3.20}$$

where the constant C indeed does not depend on $z, \tilde z, \zeta$.

If (3.10) is also satisfied by f and $\varepsilon > 0$ is arbitrary, then we can use Remark 3.6. This allows a bound where ε is multiplying the right-hand side of (3.20). □

We remind that $\widehat{f}_r \in$ BO or $\widehat{f}_r \in$ VO whenever $f \in {\rm BMO}^1$ or $f \in {\rm VMO}^1$, respectively (see (2.4)). We improve on these in the following theorem, which is the main result of this section. See Remark 3.3 for some relevant explanations, and also (2.4).

Theorem 3.8. *If f satisfies the* BWMO*-condition (3.9), then the function $\widehat{f}$ (see (3.8)) belongs to* BO*, and if f satisfies the* VWMO*-condition (3.10), then $\widehat{f} \in$* VO.

Proof. Let us again fix $z \in \mathbb{D}$ and consider w such that hyperbolic distance $\beta(z,w) \leq \delta$; by choosing $\delta > 0$ small enough (independently of z, w) it is clear that the intersection $B(z) \cap B(w)$ can be written as $B(\tilde{z},\zeta)$ and $B(\tilde{w},\xi)$ for some points $\tilde{z}, \zeta \in B(z)$ and $\tilde{w}, \xi \in B(w)$ such that both

$$\frac{|B(z)|}{|B(\tilde{z},\zeta)|} \leq 2 \text{ and } \frac{|B(w)|}{|B(\tilde{w},\xi)|} \leq 2. \tag{3.21}$$

For example, if $z = re^{i\theta}$ and $w \in B(z)$, we choose $\tilde{z} = \tilde{w} = w$ and

$$\zeta = \xi = \big(1 - 2^{-1}(1-r)\big)e^{i(\theta + \pi(1-r))},$$

see the definition of the set $B(z)$ in (3.1). We get

$$|\widehat{f}(z) - \widehat{f}(w)| \leq |\widehat{f}(z) - \widehat{f}_{B(\tilde{z},\zeta)}| + |\widehat{f}_{B(\tilde{w},\xi)} - \widehat{f}(w)| \tag{3.22}$$

and both of these terms can be bounded by a constant times $\|f\|_{\mathrm{BWMO}}$, by Corollary 3.7. The second claim follows then from the last part of Corollary 3.7. □

4. Applications to spectra and Fredholm properties

In order to characterize the essential spectra of Toeplitz operators with symbols satisfying the VWMO condition, we recall that the Berezin transform $\widetilde{T}$ of a bounded linear operator $T : A^p \to A^p$ is given by

$$\widetilde{T}(z) := \frac{\langle TK_z, K_z\rangle_{L^2}}{\langle K_z, K_z\rangle_{L^2}}, \quad z \in \mathbb{D},$$

which is well-defined because $K_z \in A^p$ for all $p \in (1,\infty)$ (cf. [13], Ch. 6). Moreover, $\widetilde{T}$ is always a bounded continuous function on $\mathbb{D}$. For $f \in L^1$ such that the Toeplitz operator T_f is bounded in A^2 we get $\widetilde{T_f} = \widetilde{f}$; see [13], p. 165. We will need a generalization of this for each function f that satisfies (3.4), that is, there exists a constant $C > 0$ such that $|\widehat{f}(z,\zeta)| \leq C$ for all $z \in \mathbb{D}$ and all $\zeta \in B(z)$.

Definition 4.1. If $f \in L^1_{\mathrm{loc}}(\mathbb{D})$ satisfies (3.4), then we define its Berezin transform by $\widetilde{f} := \widetilde{T_f}$.

Recall that condition (3.4) implies the boundedness of T_f in the space A^2, hence, the definition coincides with the conventional one in the case mentioned above. We note that the Berezin transform is again independent of p. In the following, we still denote the unique extension of the Berezin transform of f to $\beta\mathbb{D}$ by $\widetilde{f}$. Let $C_\partial(\mathbb{D})$ denote the set of continuous functions on $\mathbb{D}$ that have zero limits at the boundary (equivalently: continuous functions on $\beta\mathbb{D}$ that vanish on $\beta\mathbb{D} \setminus \mathbb{D}$).

The following corollary includes generalizations of a number of known results to new symbol classes. Concrete symbols satisfying the assumptions of the corollary are considered in Example 4.8. Estimates for the essential norm of operators in the Toeplitz algebra on A^p were previously obtained by Suárez [7] and on weighted Bergman spaces A^p by Mitkovski, Suárez and Wick [4], which were further improved and extended to bounded symmetric domains in [2]. We emphasize that the essential norm was previously computed exactly only when $p = 2$, while it remains an open problem to find a sharp constant for the upper estimate for the other values of p. Part (b) of Corollary 4.2 was proved in [15] for symbols f in BMO^1 that satisfy the condition $\tilde{f} \in L^\infty \cap VO$ (or equivalently, $\hat{f} \in L^\infty \cap VO$).

Corollary 4.2. *Assume that the symbol $f \in L^1_{\mathrm{loc}}(\mathbb{D})$ satisfies* VWMO-*condition* (3.10) *and that the average function $\widehat{f}$ belongs to L^∞. Then, the Toeplitz operator $T_f : A^p \to A^p$ is bounded for all $1 < p < \infty$ and there holds*

(a) $\widetilde{f} - \widehat{f} \in C_\partial(\mathbb{D})$,

(b) $\mathrm{spec}_{\mathrm{ess}}(T_f) = \widehat{f}(\beta\mathbb{D} \setminus \mathbb{D}) = \widetilde{f}(\beta\mathbb{D} \setminus \mathbb{D})$,

(c) $\max\limits_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\widetilde{f}(y)| = \max\limits_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\widehat{f}(y)| \le \|T_f\|_{\mathrm{ess}} \le \|P\| \max\limits_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\widehat{f}(y)|$,

(d) $\|T_f\|_{\mathrm{ess}} = \max\limits_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\widehat{f}(y)| = \max\limits_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\widetilde{f}(y)|$ *for* $p = 2$.

Moreover, T_f is Fredholm if and only if $T_{\widehat{f}}$ is if and only if $T_{\widetilde{f}}$ is.

Proof. By Theorem 3.8 we have $\widehat{f} \in \mathrm{VO}$ which clearly implies that

$$\lim_{|z|\to 1} \frac{1}{|B(z)|} \sup_{\zeta \in B(z)} \Big| \int\limits_{B(z,\zeta)} \big(\widehat{f} - \widehat{f}(z)\big) dA \Big| = 0. \tag{4.1}$$

Combining this with the assumption that f satisfies (3.10) we see that the function $f - \widehat{f}$ satisfies condition (3.6). This implies that the Toeplitz operator $T_{f-\widehat{f}} = T_f - T_{\widehat{f}}$ is compact while $T_{\widehat{f}}$ is bounded by assumption. It also implies that f satisfies (3.4), which means that $\widetilde{f}$ is well-defined, by Definition 4.1. Further, by [11], we have $\widehat{f} - \widetilde{\widetilde{f}} \in C_\partial(\mathbb{D})$ and

$$\widetilde{f} - \widehat{f} = \widetilde{\widehat{f}} - \widehat{f} + (f - \widehat{f})^\sim \in C_\partial(\mathbb{D}),$$

where $(f - \widehat{f})^\sim \in C_\partial(\mathbb{D})$ follows from [7, Theorem 9.5]. This proves (a).

Also, the case (b) follows from the above observations and Theorem 2.1. Moreover, we find that it is enough to prove (c) and (d) for the function $\widehat{f}$ instead of f. Hence, by Theorem 3.8 and a redefinition of the notation, we may assume that $f \in L^\infty \cap \mathrm{VMO}^1$ for the rest of the proof. First, for all $z \in \mathbb{D}$ we denote by $U_z : L^p \to L^p$ the surjective isometry (reflection)

$$(U_z f)(w) = f(\phi_z(w)) \frac{(1 - |z|^2)^{2/p}}{(1 - z\bar{w})^{4/p}}, \quad w \in \mathbb{D}. \tag{4.2}$$

Notice that $U_z^{-1} = U_z$ and $U_z M_f U_z^{-1} = M_{f\circ\phi_z}$, where M_f is the multiplication operator defined by $M_f g = fg$.

Let $z \in \beta\mathbb{D}$ and choose a net (z_γ) in $\mathbb{D}$ that converges to z. We note that the operator $U_{z_\gamma} T_f U_{z_\gamma}^{-1}$ converges strongly to T_{h_z}, where h_z is a bounded and analytic function on $\mathbb{D}$. This follows from Lemma 2.1 and Proposition 2.2 of [3]. By Theorem 22 and formula (4.1) of [2], we have

$$\sup_{z \in \beta\mathbb{D}\setminus\mathbb{D}} \|T_{h_z}\| \le \|T_f\|_{\text{ess}} \le \|P\| \sup_{z \in \beta\mathbb{D}\setminus\mathbb{D}} \|T_{h_z}\|.$$

As h_z is bounded and analytic, we have $\|T_{h_z}\| \le \|h_z\|_\infty$. Moreover, it is well-known that $\operatorname{spec}(T_{h_z}) = \operatorname{clos}(h_z(\mathbb{D}))$ ("clos" denotes the closure of the set). Indeed, $T_{(h_z-\lambda)^{-1}}$ is an inverse of $T_{h_z-\lambda}$ if $\lambda \notin \operatorname{clos}(h_z(\mathbb{D}))$ and conversely $\overline{h_z(w)}$ is an eigenvalue of $T^*_{h_x}$ for every $w \in \mathbb{D}$ since

$$T^*_{h_z} K_w = P(\overline{h_z} K_w) = \overline{h_z(w)} K_w.$$

Since $\|T_{h_z}\| \ge w$ for all $w \in \operatorname{spec}(T_{h_z})$, we get $\|T_{h_z}\| = \|h_z\|_\infty$. Moreover,

$$\lim_{z_\gamma \to x} \tilde{f}(\phi_{z_\gamma}(w)) = \lim_{z_\gamma \to x} \widetilde{f \circ \phi_{z_\gamma}}(w) = \widetilde{h_x}(w) = h_x(w)$$

implies

$$\sup_{x \in \beta\mathbb{D}\setminus\mathbb{D}} \|h_z\|_\infty = \max_{y \in \beta\mathbb{D}\setminus\mathbb{D}} |\tilde{f}(y)|.$$

Combining these estimates, we obtain (c) and (d). □

Corollary 4.3. *If $f \in L^1(\mathbb{D})$ satisfies the* VWMO*-condition* (3.10)*, $\widehat{f}$ is bounded and T_f is Fredholm on A^p for $1 < p < \infty$, then*

$$\operatorname{ind} T_f = \operatorname{ind} T_{\widehat{f}} = -\operatorname{wind} \widehat{f}|_{\{|z|=r\}}$$

where wind *denotes the winding number and r is sufficiently close to* 1*. The same statement also holds if $\widehat{f}$ is replaced by $\widetilde{f}$.*

Proof. Notice first that

$$\operatorname{ind} T_f = \operatorname{ind} T_{\widehat{f}} + \operatorname{ind} T_{f-\widehat{f}} = \operatorname{ind} T_{\widehat{f}}$$

because $T_{f-\widehat{f}}$ is compact. By Theorem 3.8, $\widehat{f} \in \mathrm{VO}$, and the rest now follows from Theorem 2.1 (ii) and Corollary 4.2. □

In the next result we need to assume $f \in L^1(\mathbb{D})$ instead of $f \in L^1_{\text{loc}}(\mathbb{D})$, although some implications would hold even for locally integrable symbols.

Corollary 4.4. *Assume that the symbol $f \in L^1(\mathbb{D})$ satisfies* BWMO*-condition* (3.9)*. Then, the following are equivalent:*

(i) $T_f : A^p \to A^p$ *is bounded for some* $1 < p < \infty$*,*

(ii) $T_f : A^p \to A^p$ *is bounded for all* $1 < p < \infty$*,*

(iii) $\widehat{f} \in L^\infty(\mathbb{D})$*,*

(iv) $\widetilde{f} \in L^\infty(\mathbb{D})$*.*

Proof. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (iv): Assume (iii) holds. We have, for some constant $C > 0$ and all $z \in \mathbb{D}$ and $\zeta \in B(z)$,

$$\frac{1}{|B(z)|}\Big|\int_{B(z,\zeta)} (f - \widehat{f})dA\Big| \le \frac{1}{|B(z)|}\Big|\int_{B(z,\zeta)} \big(f(\xi) - \widehat{f}(z)\big)dA(\xi)\Big|$$
$$+\frac{1}{|B(z)|}\Big|\int_{B(z,\zeta)} \big(\widehat{f}(\xi) - \widehat{f}(z)\big)dA(\xi)\Big| \le C, \tag{4.3}$$

because the first integral on the right is bounded due to $f \in \mathrm{BWMO}$ and the second one due to $\widehat{f} \in L^\infty$. Hence, the function $f - \widehat{f}$ satisfies condition (3.4), and thus the Toeplitz operator $T_{f-\widehat{f}}$ is bounded in any A^p. The boundedness of the average function $\widehat{f}$ also implies that the operator $T_f = T_{f-\widehat{f}} + T_{\widehat{f}}$ is bounded in any A^p. The boundedness of T_f yields the boundedness of the Berezin transform $\widetilde{f}$, see the beginning of Section 4.

(i) $\Rightarrow$ (iii): Assume p is such that T_f is bounded in A^p. We can use (4.3) to see that the operator $T_{f-\widehat{f}}$ is bounded in A^p. The boundedness of the second integral on the right of (4.3) follows from $\widehat{f} \in \mathrm{BO}$, see Theorem 3.8. Next, T_f and $T_{f-\widehat{f}}$ bounded implies $T_{\widehat{f}}$ bounded, hence the Berezin transform $\widetilde{\widehat{f}}$ is a bounded function, and this fact does not depend on p. Since $\widehat{f} \in \mathrm{BO} \subset \mathrm{BMO}^1$ we obtain that also $\widehat{\widehat{f}} \in L^\infty(\mathbb{D})$ (see Corollary 2.3.(d) of [14]). Also, $\widehat{f} \in \mathrm{BO}$ implies $\big(f - \widehat{f}\big)^\wedge \in L^\infty(\mathbb{D})$. We obtain

$$\widehat{f} = \big(f - \widehat{f}\big)^\wedge + \widehat{\widehat{f}} \in L^\infty(\mathbb{D}).$$

(iv) $\Rightarrow$ (iii) It suffices to show that $\widetilde{f} \in L^\infty$ implies $\widetilde{\widehat{f}} \in L^\infty$. Once we have this, we deduce as in the previous item that $\widehat{\widehat{f}} \in L^\infty(\mathbb{D})$ and then $\big(f - \widehat{f}\big)^\wedge \in L^\infty(\mathbb{D})$ and finally $\widehat{f} \in L^\infty(\mathbb{D})$.

We utilize the decomposition of the disc into the hyperbolic sets $B_n = B(z_n)$, mentioned in Remark 3.2 (ii) and write

$$\widetilde{\widehat{f}}(z) = \int_{\mathbb{D}} \widehat{f}|k_z|^2 dA = \sum_{n\in\mathbb{N}} \int_{B_n} \widehat{f}(w)|k_z(w)|^2 dA(w). \tag{4.4}$$

We note that since $\widehat{f} \in \mathrm{BO}$, the modulus of the expression

$$\widehat{f}(w) - \widehat{f}(z_n) =: F_n(w)$$

is bounded for every $w \in B_n$ by a constant $C_1 > 0$ independent of w or n. Hence, (4.4) equals

$$\sum_{n\in\mathbb{N}} \int_{B_n} \widehat{f}(z_n)|k_z(w)|^2 dA(w) + \sum_{n\in\mathbb{N}} \int_{B_n} F_n|k_z|^2 dA, \tag{4.5}$$

where

$$\Big|\sum_{n\in\mathbb{N}}\int_{B_n} F_n|k_z|^2dA\Big| \le \sum_{n\in\mathbb{N}}\int_{B_n} C_1|k_z|^2dA = C_1\|k_z\|_2^2 = C_1$$

since k_z is the normalized kernel, see (2.5). To estimate the first term in (4.5) we write

$$G_z(w,\zeta) = |k_z(w)|^2 - |k_z(\zeta)|^2, \tag{4.6}$$

which again is bounded by a constant $C_2 > 0$ independent of $z \in \mathbb{D}$ and $n \in \mathbb{N}$ and $w, \zeta \in B_n$. We get

$$\sum_{n\in\mathbb{N}}\int_{B_n} \widehat{f}(z_n)|k_z(w)|^2dA(w) = \sum_{n\in\mathbb{N}}\int_{B_n} \frac{1}{|B_n|}\int_{B_n} f(\zeta)|k_z(\zeta)|^2dA(\zeta)dA(w)$$

$$+\sum_{n\in\mathbb{N}}\int_{B_n} \frac{1}{|B_n|}\int_{B_n} f(\zeta)G_z(w,\zeta)dA(\zeta)dA(w). \tag{4.7}$$

Here, the first term equals $\widetilde{f}$. The second one is bounded by

$$\sum_{n\in\mathbb{N}}\int_{B_n} \frac{1}{|B_n|}\int_{B_n} C_2|f(\zeta)|dA(\zeta)dA(w) \le C_2\sum_{n\in\mathbb{N}}\int_{B_n} |f(\zeta)|dA(\zeta) \le C_2\|f\|_1.$$

This completes the proof of the corollary. □

Corollary 4.5. (i) *Assume that the symbol $f \in L^1(\mathbb{D})$ satisfies* BWMO-*condition* (3.9)*. Then, the Toeplitz operator $T_{f-\widehat{f}}$ is bounded in A^p for all $1 < p < \infty$.*

(ii) *If $f \in L^1(\mathbb{D})$ satisfies* VWMO-*condition* (3.10)*, then $T_{f-\widehat{f}}$ is compact in A^p for all $1 < p < \infty$. If, in addition, T_f is bounded in A^p for some $1 < p < \infty$, then also $T_{f-\widetilde{f}}$ is compact in all spaces A^p.*

Proof. The boundedness of $T_{f-\widehat{f}}$ follows from the proof of Corollary 4.4, (i) ⇒ (iii), and the compactness from the beginning of the proof of Corollary 4.2. If in addition T_f is bounded, Corollary 4.4 yields that $\widehat{f} \in L^\infty$. We obtain from Corollary 4.2 that the bounded, continuous function $\widetilde{f} - \widehat{f}$ belongs to $C_\partial(\mathbb{D})$. Hence, the operator $T_{\widetilde{f}-\widehat{f}}$ and thus also $T_{f-\widetilde{f}}$ are both compact. □

Remark 4.6. (i) The proof of Theorem 3.7 in [15] shows that if $f \in \mathrm{BMO}^2$ and the Berezin transform $\widetilde{f} \in L^\infty \cap \mathrm{VO}$, then the Toeplitz operator $T_{f-\widetilde{f}}$ is compact on A^p. Since in this case $\widetilde{f}$ is a bounded continuous function, the results concerning the Berezin transform in (b)–(d) of Corollary 4.2 hold true for such symbols.

(ii) It is known that for $f \in \mathrm{BMO}^1$, the compactness of $T_f : A^p \to A^p$ for some $1 < p < \infty$ is equivalent to the vanishing of the Berezin transform at the boundary, i.e. $\widetilde{f} \in C_\partial(\mathbb{D})$; see [14], Theorem 3.1. We do not know if this result can be generalized for symbols in the BWMO-class.

We give one simple application of Theorem 3.8 to the study of block Toeplitz operators. For a suitable $f = (f_{jk})_{j,k=1}^N$ with $f_{jk} \in L^1$, $N \geq 2$, denote the block Toeplitz operator by $T_f : A_N^p \to A_N^p$, where $A_N^p = \{g = (g_1, \dots, g_N) : g_k \in A^p\}$ with $\|g\|_{A_N^p} = \max \|g_k\|_p$. More precisely, if

$$g = (g_1, \dots, g_N) = (g_k)_{k=1,\dots,N} \in A_N^p,$$

then

$$T_f g = P(fg^T) = P\left(\left(\sum_{k=1}^N f_{jk} g_k\right)_{j=1,\dots,N}\right) = \left(\sum_{k=1}^N T_{f_{jk}} g_k\right)_{j=1,\dots,N},$$

where g^T is the transpose of g. Similarly, the Berezin transform of T_f is a matrix operator which is defined componentwise by using the scalar definition in the beginning of Section 4. For further details, see [5]

For scalar symbols $f, g \in L^\infty$, it is easy to see that

$$T_f T_g = T_{fg} - PM_f H_g. \tag{4.8}$$

Note that the Hankel operator H_g is compact in any L^p if $g \in \mathrm{VO}$, see, e.g., [3].

Proposition 4.7. *Let $f = (f_{jk})$ with $f_{jk} \in L^1 \cap \mathrm{VWMO}$ and suppose that $\widehat{f}_{jk} \in L^\infty$. Then T_f is Fredholm on A_N^p if and only if $\det \widetilde{f}$ is bounded away from zero near $\partial\mathbb{D}$.*

Proof. First, we note that all operators $T_{jk} : A^p \to A^p$ and $T_f : A_N^p \to A_N^p$ are bounded as a consequence of Corollary 4.4. Hence, the Berezin transform $\widetilde{f} = (\widetilde{f}_{jk})$ is a well defined, bounded and continuous matrix function $\mathbb{D} \to \mathbb{C}^{N\times N}$. Moreover, $T_{f-\widetilde{f}}$ is compact on A_N^p by Corollary 4.5.

We reduce the matrix-valued case to the scalar case using the following well-known theorem: if the entries A_{jk} of a bounded linear matrix operator A on a product Banach space X^N commute modulo compact operators, then A is Fredholm on X^N if and only if $\det A$ is Fredholm on X (see, e.g., [5]).

Since

$$T_f = T_{\widetilde{f}} + T_{f-\widetilde{f}}$$

and $T_{f-\widetilde{f}}$ is compact on A_N^p, it follows that T_f is Fredholm if and only if $T_{\widetilde{f}}$ is Fredholm. By (4.8), if the scalar symbols g, h belong to $L^\infty \cap \mathrm{VO}$, then T_g and T_h commute modulo compact operators. Now, by Theorem 3.8, all $\widehat{f}_{jk}$ belong to $L^\infty \cap \mathrm{VO}$, and the same is true also for all $\widetilde{f}_{jk}$, by Corollary 4.2.(a). We conclude by the above mentioned theorem that $T_{\widetilde{f}}$, equivalently, T_f, are Fredholm if and only if $\det T_{\widetilde{f}}$ is Fredholm. Notice that

$$\det T_{\widetilde{f}} = \sum_{\sigma \in S_N} \left(\operatorname{sgn}(\sigma) \prod_{j=1}^N T_{\widetilde{f}_{j,\sigma_j}}\right) = T_{\det \widetilde{f}} + K,$$

where S_N is the permutation group and K is some compact operator. Therefore, because $\widetilde{f}_{jk} \in L^\infty \cap \mathrm{VO}$ so that $\det \widetilde{f} \in L^\infty \cap \mathrm{VO}$, the scalar case (see Corollary 4.2 or [1, 5]) implies that T_f is Fredholm if and only if $\det \widetilde{f}$ is bounded away from zero near $\partial\mathbb{D}$. □

Example 4.8. We present examples of symbols which (i) satisfy the VWMO-condition (3.10) but which are not in BMO^1, or (ii) bounded VWMO-symbols which are not in VMO^1. In the following examples the average function $\widehat{f}$ belongs to $C_\partial(\mathbb{D})$ and the operator $T_{\widehat{f}}$ is compact. Although the example resembles that in Remark 2.4 of [8], the proof is completely different, the reason being that the standard definition of BMO^1 involves hyperbolic discs $D(z,r)$ instead of the sets $B(z)$, and the present technique is more convenient here. We define for all $b \geq \beta > 0$ the function

$$f(re^{i\theta}) := \begin{cases} \dfrac{1}{r(1-r)^{b-\beta}} \sin \dfrac{1}{(1-r)^b}\,, & r \geq \frac{1}{2} \\ 1\,, & r < \frac{1}{2}. \end{cases} \tag{4.9}$$

Given $z = re^{i\theta}$ we calculate using integration by parts, for all $\zeta = \tilde{r}e^{i\tilde{\theta}} \in B(z)$, $\xi = \rho e^{i\phi} \in B(z,\zeta)$,

$$\int\limits_{B(z,\zeta)} f(\xi)\, dA(\xi) = \int\limits_{\theta}^{\tilde{\theta}} \int\limits_{r}^{\tilde{r}} (1-\rho)^{1+\beta} \frac{1}{(1-\rho)^{b+1}} \sin \frac{1}{(1-\rho)^b} d\rho d\phi$$

$$= \ -\int\limits_{\theta}^{\tilde{\theta}} \Big(\Big[\frac{(1-\rho)^{1+\beta}}{b} \cos \frac{1}{(1-\rho)^b} \Big]_{\rho=r}^{\rho=\tilde{r}}$$

$$+\frac{1+\beta}{b} \int\limits_{r}^{\tilde{r}} (1-\rho)^{\beta} \cos \frac{1}{(1-\rho)^b} d\rho \Big) d\phi \tag{4.10}$$

We estimate $|\cos(...)| \leq 1$ and take into account the lengths of the integration intervals, hence, the modulus of the replacement term is bounded by

$$C \int\limits_{\theta}^{\tilde{\theta}} (1-r)^{1+\beta} d\phi \leq C'(1-r)^{2+\beta}$$

and the last integral in (4.10) has the same bound. Since $|B(z)| \geq C(1-r)^2$, we obtain the following two conclusions: first, f satisfies condition (3.6) and second, $|\widehat{f}| \in C_\partial(\mathbb{D})$. These two imply $f \in \mathrm{VWMO}$.

To see that $f \notin \mathrm{BMO}^1$ we first show that also the average function $\widehat{f}_1$, see (2.2), belongs to $C_\partial(\mathbb{D})$. To this end we fix $z = re^{i\theta}$ and consider a set $D(z,1)$

instead of $B(z)$. It follows from the definition of the hyperbolic geometry, see [13], Ch. 6, that for some constant $C > 0$ we have for all $\zeta \in D(z,1)$

$$\begin{aligned} &\frac{1}{C}(1-r) \le 1-|\zeta| \le C(1-r), \\ &\sup_{w_1,w_2\in D(z,1)} |w_1-w_2| \le C(1-r). \end{aligned} \tag{4.11}$$

It is also obvious that the set $D(z,1)$ can be presented using polar coordinates as

$$D(z,1) \;=\; \big\{\xi = \rho e^{i\phi} \;:\; \theta-\theta_0 < \phi < \theta+\theta_0,\ r_1(\phi) < \rho < r_2(\phi)\big\}$$

for some number $\theta_0 > 0$ and functions $r_j : (\theta-\theta_0, \theta+\theta_0) \to (0,1)$. The points $r_j(\phi)e^{i\phi}$, $j = 1,2$, form the boundary of the disc $D(z,1)$; moreover, by (4.11), θ_0 and r_j are bounded by $C(1-r)$ and also bounded from below by $C'(1-r)$.

We obtain in the same way as in (4.10)

$$\begin{aligned} \int\limits_{D(z,1)} f(\xi)\,dA(\xi) = \int\limits_{\theta-\theta_0}^{\theta+\theta_0} \bigg(\Big[\frac{(1-\rho)^{1+\beta}}{b} \cos\frac{1}{(1-\rho)^b} \Big]_{\rho=r_1(\phi)}^{\rho=r_2(\phi)} \\ + \frac{1+\beta}{b} \int\limits_{r_1(\phi)}^{r_2(\phi)} (1-\rho)^\beta \cos\frac{1}{(1-\rho)^b}\,d\rho \bigg) d\phi, \end{aligned}$$

and as above we see that the modulus of this is bounded by $C(1-r)^{2+\beta}$. Hence, $|\widehat{f}_1(z)| \le C(1-r)^\beta$ for all $z = re^{i\theta} \in \mathbb{D}$ and in particular $\widehat{f}_1 \in C_\partial(\mathbb{D})$.

It is quite obvious that there is a constant $\delta > 0$ such that we have the lower bound $|\sin(1/(1-r))| \ge \delta$ for $re^{i\theta}$ in a subset D_3 of $D(z,1)$ with measure at least $|D(z,1)|/2$, thus $|f(z)| \ge C(1-r)^{\beta-b}$ for $z = re^{i\theta} \in D_3$. As a consequence

$$\begin{aligned} &\frac{1}{|D(z,1)|} \int\limits_{D(z,1)} |f(\xi) - \widehat{f}_1(z)|\,dA(\xi) \\ \ge\;& \frac{1}{|D(z,1)|} \int\limits_{D_3} |f|\,dA - \frac{1}{|D(z,1)|} \int\limits_{D_3} |\widehat{f}_1(z)|\,dA(\xi) \\ \ge\;& \frac{C}{|D(z,1)|} \int\limits_{D_3} \frac{1}{(1-r)^{b-\beta}}\,dA - \frac{C'}{|D(z,1)|} \int\limits_{D_3} (1-r)^\beta\,dA \ge C''(1-r)^{\beta-b}. \end{aligned}$$

In view of the definition of the norm of BMO^1, see (2.1), we get an example of type (i) by taking any parameters b, β such that $b > \beta > 0$. Moreover, if $b-\beta < 1$, then there holds $f \in L^1$, but if $b-\beta \ge 1$, we only have $f \in L^1_{\mathrm{loc}}$. To obtain an example of type (ii) one chooses $b = \beta > 0$.

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